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A Stability Result for the Solutions of a Certain Fourth-Order Vector Differential Equation with Delay

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In this paper, by constructing an appropriate Lyapunov functional, we establish sufficient conditions for the uniform stability of the zero solution for nonlinear fourth-order vector delay differential equation of the type:

$$\boldsymbol{X}^{(4)} + \boldsymbol{F}(\dddot{\boldsymbol{X}}) + \boldsymbol{\Psi}(\dot{\boldsymbol{X}})\ddot{\boldsymbol{X}} + \boldsymbol{G}(\dot{\boldsymbol{X}}(\boldsymbol{t}-\boldsymbol{r})) + \boldsymbol{H}(\boldsymbol{X}(\boldsymbol{t}-\boldsymbol{r})) = \boldsymbol{0}.$$

The obtained results included improve some well-known results existing in the related literature. An example is given to illustrate the truthfulness of our main result.

Keywords and phrases*:* Uniform stability; Lyapunov functional; Delay vector differential equation of fourth-order.

1 INTRODUCTION

In mathematical literature, stability of solutions receives broad attention from researchers because it plays a fundamental role in the qualitative theory and applications of differential equations. Many methods have been improved to obtain information on the stability behavior of differential equations when there are no analytical formulas for the solutions. One of the most interesting methods to determine the stability behavior for the solutions of linear and nonlinear differential equations is a method known as Lyapunov's second (or direct) method [7]. The main advantage of this method is that stability behavior can be obtained without any previous knowledge for solutions. That is, this method gives stability information directly, without solving the differential equation.

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Today, this method is considered as an effective tool not only in the study of the stability of solutions for differential equations but also in the theory of control systems, analysis of energy system, dynamic systems, systems with time lag, and so on. It should be noted that any verification on the stability of solutions for vector functional differential equations of fourth-order, using the Lyapunov functional method, first requires construction of a suitable Lyapunov functional. In fact, the constructing of an appropriate Lyapunov functional is in mostly a difficult work.

Over the past years, many new results have been obtained on the stability for solutions of ordinary and functional differential equations of higher order without and with delay. For instance, we draw the attention of the interested reader to the book by Reissig et al. [10] and the papers by Abou El-Ela et al. [1, 2, 3], Adesina et al. [4], Omeike [8, 9], Sadek [11], Tunç [12, 13, 14] and the references cited therein. As far as we know, researches that discussed the stability of solutions to vector differential equations can briefly be summarized as follows:

First, in 2006 Tunç [13] gave sufficient conditions for the asymptotic stability of the trivial solution $X = 0$ of equation:

$$X^{(4)} + \Phi(\ddot{X})\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = 0,$$

Where $X \in \mathcal{R}^n$; F and Φ are $n \times n$-symmetric matrices; G and H are n-vector continuous functions; $G(0) = H(0) = 0$.

After that, in 2012 Abou-El-Ela et al. [2] established sufficient conditions for the uniform stability of the zero solution of the real fourth-order vector delay differential equation:

$$X^{(4)} + A\dddot{X} + \Phi(\ddot{X}) + G(\dot{X}) + H(X(t-r)) = 0,$$

where $X \in \mathcal{R}^n$; A is continuous $n \times n$-symmetric matrix; Φ, G and H are n-vector continuous functions; $\Phi(0) = G(0) = H(0) = 0$; r is a fixed delay and positive constant.

Lately, in 2015 Abou-El-Ela et al. [3] investigated sufficient conditions for the uniform stability of the zero solution $X = 0$ of real nonlinear autonomous vector delay differential equation of the fourth-order:

$$X^{(4)} + F(X, \dot{X})\dddot{X} + \Phi(\ddot{X}) + G(\dot{X}(t-r)) + H(X(t-r)) = 0,$$

where $X \in \mathcal{R}^n$; F is an $n \times n$-symmetric matrix; Φ, G and H are n-vector continuous functions; $\Phi(0) = G(0) = H(0) = 0$; r is a bounded delay and positive constant.

The objective of this paper is to study the uniform stability of the zero solution of vector delay differential equation of the form:

$$X^{(4)} + F(\dddot{X}) + \Psi(\dot{X})\ddot{X} + G\left(\dot{X}(t-r)\right) + H\big(X(t-r)\big) = 0, \quad (1.1)$$

where r is the fixed delay and positive constant; $X \in \mathcal{R}^n$; Ψ is an $n \times n$ continuous symmetric matrix function; F, G and H are n-vector continuous functions; $F(0) = G(0) = H(0) = 0$.

It should be noted that the continuity of functions F, Ψ, G and H is a sufficient condition for existence of the solution of (1.1). In addition, we assume that the functions F, Ψ, G and H satisfy a Lipschitz condition with respect to X, $\dot{X}$, $\ddot{X}$ and $\dddot{X}$, this assumption is guaranteed the uniqueness of solution of (1.1).

Equation (1.1) can be represented as a system of real fourth-order delay differential equations:

$$x_i^{(4)} + f_i(\dddot{x}_1, \ldots, \dddot{x}_n) + \sum_{k=1}^{n} \psi_{ik}(\dot{x}_1, \ldots, \dot{x}_n)\ddot{x}_k + g_i(\dot{x}_1(t-r), \ldots, \dot{x}_n(t-r))$$
$$+ h_i\big(x_1(t-r), \ldots, x_n(t-r)\big) = 0, \qquad (i = 1, 2, \ldots, n).$$

Let $J_F(W)$, $J_G(Y)$, $J_H(X)$ and $J(\Psi(Y)Y|Y)$ denote the Jacobian matrices corresponding to the functions $F(W)$, $G(Y)$, $H(X)$ and the matrix $\Psi(Y)$ respectively which given by the following relations:

$$J_F(W) = \left(\frac{\partial f_i}{\partial w_j}\right), \; J_G(Y) = \left(\frac{\partial g_i}{\partial y_j}\right), \; J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right) \qquad \text{and}$$
$$J(\Psi(Y)Y|Y) = \frac{\partial}{\partial y_j}\left(\sum_{k=1}^{n} \psi_{ik} y_k\right) = \Psi(Y) + \left(\sum_{k=1}^{n} \frac{\partial \psi_{ik}}{\partial y_j} y_k\right),$$

where x_i, $y_i = \dot{x}_i$, $z_i = \ddot{x}_i = \dot{y}_i$, $w_i = \dddot{x}_i = \dot{z}_i$, f_i, ψ_{ij}, g_i and h_i $(i, j = 1, \ldots, n)$, represent X, Y, Z, W, F, Ψ, G and H respectively.

In the following, we assume that the Jacobian matrices, $J_H(X)$, $J_G(Y)$,

$J(\Psi(Y)Y|Y)$ and $J_F(W)$ exist and are continuous. Besides, the symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\mathcal{R}^n$ denoted to the usual scalar product in $\mathcal{R}^n$, that is $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X\rangle = \| X \|^2$, $\lambda_i(A)$ $(i = 1,2, \dots, n)$ are the eigenvalues of the $n \times n$ matrix A.

2 Main Result

To reach the main result of this paper, we will offer some essential information to the stability criteria for a general autonomous delay differential system. We consider

$$\dot{\bar{x}} = \bar{f}(\bar{x}_t), \quad \bar{x}_t(\theta) = (t+\theta), \quad -r \le \theta \le 0, \ t \ge 0. \tag{2.1}$$

where $\bar{f}: C_H \to \mathcal{R}^n$ is a continuous, $\bar{f}$ takes closed bounded sets into bounded sets, $C_H := \{\phi \in C([-r, 0], \mathcal{R}^n) : \|\phi\| < H\}$, $\bar{f}(0) = 0$ and for $H_1 < H$, there exists $L(H_1) > 0$, with $|\bar{f}(\phi)| \le L(H_1)$ when $\|\phi\| < H_1$.

Theorem 2.1. [6] Assume that there exists a continuous functional $V(\phi): C_H \to \mathcal{R}$ satisfying a local Lipschitz condition, $V(0) = 0$, such that:
(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\| \phi \|)$, where W_1, W_2 are wedges and
(ii) $\dot{V}_{(2.1)}(\phi) \le 0$, for $\phi \in C_H$.
Then the zero solution of equation (2.1) is uniformly stable.

Now we will present our main stability result of (1.1) as the following:
Theorem 2.2. Beside the basic assumptions which put on the functions F, Ψ, G and H, we assume that there exist positive constants $\alpha_1, \alpha_2, \alpha_3$, α_4 and α_4' such that for $(i = 1, 2, 3, \dots, n)$ the following conditions are hold:
(i) $\Psi(Y)$ *is symmetric and*

$$0 \le \lambda_i(\Psi(Y) - \alpha_2 I) \le \frac{1}{4}\alpha_1^3 \varepsilon, \ for\ all\ Y \in \mathcal{R}^n.$$

(ii) $G(0) = 0$, $J_G(Y)$ *is symmetric and*

$$\lambda_i\left(\int_0^1 J_G(\sigma Y) d\sigma\right) \ge \frac{1}{2}\alpha_3, \ for\ all\ Y \in \mathcal{R}^n.$$

(iii) *There is a finite constant* $\Delta > 0$, *such that*:

$$\{\alpha_1\alpha_2 - 4 \| J_G(Y) \| \}\alpha_3 - \alpha_1^2\alpha_4 \ge \Delta, \ for\ all\ Y \in \mathcal{R}^n.$$

(iv) $0 \le \lambda_i\left(J_G(Y) - \int_0^1 J_G(\sigma Y) d\sigma\right) \le \delta < \frac{2\Delta\alpha_4}{\alpha_1\alpha_3^2}$, $for\ all\ Y \in \mathcal{R}^n$.

(v) $H(0) = 0$, $J_H(X)$ *is symmetric and*

$$\lambda_i\left(\int_0^1 J_H(\sigma X) d\sigma\right) \ge \alpha_4', \ for\ all\ X \in \mathcal{R}^n.$$

(vi) $J_H(X)$ *commutes with* $J_H(X')$, *for all* $X, X' \in \mathcal{R}^n$ *and*

$$\lambda_i\left(\frac{1}{2}\alpha_4 I - J_H(X)\right) \ge 0, \ for\ all\ X \in \mathcal{R}^n.$$

(vii) $F(0) = 0$, $J_F(W)$ *is symmetric and*

$$0 \leq \lambda_i\left(\int_0^1 J_F(\sigma W)d\sigma - \alpha_1 I\right) \leq \frac{2\varepsilon_0\alpha_3^3}{\alpha_1\alpha_4^2}, \qquad for\ all\ W \in \mathcal{R}^n,$$

where ε_0 *is a positive constant such that*:

$$\varepsilon_0 < \varepsilon = \min\left\{\frac{1}{\alpha_1}, \frac{\alpha_4}{\alpha_3}, \frac{\Delta}{4\alpha_1\alpha_3 D_0}, \frac{\alpha_3}{4\alpha_4 D_0}\left(\frac{2\Delta\alpha_4}{\alpha_1\alpha_3^2} - \delta\right)\right\}, \tag{2.2}$$

$$and \quad D_0 = \alpha_1 + \alpha_1\alpha_2.$$

Then the zero solution of (1.1) is uniformly stable, provided that:

$$r < \min\left\{\frac{4(\varepsilon - \varepsilon_0)\alpha_3}{2\alpha_4\sqrt{n}(2d_1 + 2d_2 + 1) + \alpha_1\alpha_2 d_2\sqrt{n}}, \frac{8(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0)}{\sqrt{n}\{2\alpha_4 + \alpha_1\alpha_2(2d_1 + d_2 + 2)\}}, \frac{4(\alpha_1\varepsilon + 1)}{d_1\sqrt{n}(\alpha_1\alpha_2 + 2\alpha_4)}\right\}$$

$$where \quad d_1 = \varepsilon + \frac{1}{\alpha_1} \ \ and \ \ d_2 = \varepsilon + \frac{\alpha_4}{\alpha_3}. \tag{2.3}$$

The following lemmas are required for proving Theorem 2.2.

Lemma 2.1 [5] Let A be a real symmetric $n \times n$-matrix and

$$a' \geq \lambda_i(A) \geq a > 0 \qquad (i = 1, 2, \ldots, n),$$

where a' and a are constants. Then

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle$$

and

$$a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle.$$

Lemma 2.2 Suppose that $\dot{X} = Y$, $\dot{Y} = Z$, $\dot{Z} = W$. Then the following relations are true:

(1) $\displaystyle\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \langle H(X), Y\rangle.$

(2) $\displaystyle\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma = \langle G(Y), Z\rangle.$

(3) $\displaystyle\frac{d}{dt}\int_0^1 \langle F(\sigma W), W\rangle d\sigma = \langle F(W), \dot{W}\rangle.$

(4) $\displaystyle\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle d\sigma = \langle \Psi(Y)Z, Y\rangle.$

proof. We have

$$
\begin{aligned}
(1)\, \frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma &= \int_0^1 \sigma\langle J_H(\sigma X)Y, Xd\rangle\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma \\
&= \int_0^1 \sigma\langle J_H(\sigma X)X, Y\rangle d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma \\
&= \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle d\sigma + \int_0^1 H\langle(\sigma X), Y\rangle d\sigma \\
&= \langle H(X), Y\rangle.
\end{aligned}
$$

The proofs of (2) and (3) are similar to that of (1).

$$
\begin{aligned}
(4)\, \frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle d\sigma &= \int_0^1 \sigma\langle \Psi(\sigma Y)Y, Z\rangle d\sigma + \int_0^1 \sigma^2\langle J_\Psi(\sigma Y)YZ, Y\rangle d\sigma \\
&\quad + \int_0^1 \langle \sigma\Psi(\sigma Y)Z, Y\rangle d\sigma \\
&= \int_0^1 \sigma\langle \Psi(\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \sigma^2\langle J_\Psi(\sigma Y)YZ, Y\rangle d\sigma \\
&\quad + \int_0^1 \langle \sigma\Psi(\sigma Y)Z, Y\rangle d\sigma \\
&= \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \sigma\Psi(\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \langle \sigma\Psi(\sigma Y)Z, Y\rangle d\sigma \\
&= \langle \Psi(Y)Z, Y\rangle.
\end{aligned}
$$

3 proof of Theorem 2.2.

For the proof of Theorem 2.2, we rewrite equation (1.1) as the following equivalent system:

$$
\begin{aligned}
&\dot{X} = Y, \quad \dot{Y} = Z, \qquad \dot{Z} = W, \\
&\dot{W} = -F(W) - \Psi(Y)Z - G(Y) - H(X) + \int_{t-r}^{t} J_G(Y(s))Z(s)ds \\
&\qquad + \int_{t-r}^{t} J_H\big(X(s)\big)Y(s)ds. \qquad (3.1)
\end{aligned}
$$

proof of Theorem 2.2 needs the Lyapunov function $V = V(X_t, Y_t, Z_t, W_t)$ which is given by:

$$2\mathrm{V}(X_t, Y_t, Z_t, W_t) = 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma - d_1\langle \alpha_4 Y, Y\rangle + 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma$$
$$+2d_2 \int_0^1 \langle \Psi(\sigma Y)Y, Y\rangle d\sigma + 2d_1\langle \alpha_2 Z, Z\rangle + (\alpha_1 - d_2)\langle Z, Z\rangle + 2d_2\langle \alpha_1 Z, Y\rangle$$
$$+2d_1(\frac{\alpha_1\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0}) \int_0^1 \langle F(\sigma W), W\rangle d\sigma + d_1\langle W, W\rangle + 2H\langle (X), Y\rangle$$
$$+4d_1\langle H(X), Z\rangle + 4d_1\langle G(Y), Z\rangle + 2d_2\langle Y, W\rangle + 2\langle Z, W\rangle$$
$$+2\mu \int_{-r}^0 \int_{t+s}^t \| Y(\theta) \|^2 \, d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t \| Z(\theta) \|^2 \, d\theta ds. \quad (3.2)$$

where μ and λ are positive constants, whose values will be determined later. Let

$$\Gamma(Y) = \int_0^1 J_G(\sigma Y) d\sigma \quad (3.3)$$

then it follows from (ii) and (iv) that:

$$\lambda_i\big(\Gamma(Y)\big) \geq \frac{1}{2}\alpha_3, \; for\ all\ Y \in \mathcal{R}^n \quad (3.4)$$

$$0 \leq \lambda_i\big(J_G(Y) - \Gamma(Y)\big) \leq \delta, \; for\ all\ Y \in \mathcal{R}^n \quad (3.5)$$

Since

$$\frac{\partial}{\partial \sigma} F(\sigma W) = J_F(\sigma W)W \text{ and } F(0) = 0,$$

then

$$F(W) = \int_0^1 J_F(\sigma W)W d\sigma.$$

Therefore

$$\begin{aligned}
\int_0^1 \langle F(\sigma W), W\rangle d\sigma &= \int_0^1 \int_0^1 \langle J_F(\sigma_1\sigma_2 W)\sigma_2 W, W\rangle d\sigma_1 d\sigma_2 \\
&= \int_0^1 [\int_0^1 \langle J_F(\sigma_1 \widetilde{W})\widetilde{W}, W\rangle d\sigma_1] d\sigma_2 \\
&\geq \int_0^1 \alpha_1\langle \widetilde{W}, W\rangle d\sigma_2, \qquad by (vii) \\
&\geq \frac{1}{2}\alpha_1\langle W, W\rangle.
\end{aligned}$$

Since

$$2\mu \int_{-r}^0 \int_{t+s}^t \| Y(\theta) \|^2 \, d\theta ds \;\; and \;\; 2\lambda \int_{-r}^0 \int_{t+s}^t \| Z(\theta) \|^2 \, d\theta ds$$

are nonnegative, so we get:

$$\begin{aligned}2V(X_t,Y_t,Z_t,W_t) \geq\ & 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - d_1\langle \alpha_4 Y,Y\rangle + 2\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma \\ & +2d_2\int_0^1 \langle \Psi(\sigma Y)Y,Y\rangle d\sigma + 2d_1\langle \alpha_2 Z,Z\rangle + (\alpha_1 - d_2)\langle Z,Z\rangle \\ & +2d_2\langle \alpha_1 Z,Y\rangle + d_1(\frac{\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + 1)\langle W,W\rangle + 2\langle H(X),Y\rangle \\ & +4d_1\langle H(X),Z\rangle + 4d_1\langle G(Y),Z\rangle + 2d_2\langle Y,W\rangle + 2\langle Z,W\rangle.\end{aligned}$$

Thus we can find:

$$\begin{aligned}2V(X_t,Y_t,Z_t,W_t) \geq\ & 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - d_1\langle \alpha_4 Y,Y\rangle + 2\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma \\ & +2d_2\int_0^1 \langle \Psi(\sigma Y)Y,Y\rangle d\sigma + 2d_1\langle \alpha_2 Z,Z\rangle - d_2\langle Z,Z\rangle \\ & +d_1(\frac{\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + 1)\langle W,W\rangle - \|\Gamma^{-\frac{1}{2}}H(X)\|^2 \\ & -\|\Gamma^{\frac{1}{2}}Y\|^2 - \|2d_1\Gamma^{\frac{1}{2}}Z\|^2 - \|\alpha_1^{-\frac{1}{2}}W\|^2 - \|d_2\alpha_1^{\frac{1}{2}}Y\|^2 \\ & +\|\Gamma^{-\frac{1}{2}}H(X) + \Gamma^{\frac{1}{2}}Y + 2d_1\Gamma^{\frac{1}{2}}Z\|^2 + \|\alpha_1^{-\frac{1}{2}}W + \alpha_1^{\frac{1}{2}}Z + d_2\alpha_1^{\frac{1}{2}}Y\|^2.\end{aligned}$$

We notice that the matrix Γ defined by (3.3) is symmetric because J_G is symmetric. The eigenvalues of Γ is positive because of (3.4). Accordingly the square root $\Gamma^{\frac{1}{2}}$ exist; this is again symmetric and non-singular for all $Y \in \mathcal{R}^n$.

Therefore we have:

$$\begin{aligned}2V(X_t,Y_t,Z_t,W_t) \geq\ & 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \langle \Gamma^{-1}H(X),H(X)\rangle \\ & +2\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma - \langle \Gamma Y,Y\rangle + 2d_2\int_0^1 \langle \Psi(\sigma Y)Y,Y\rangle d\sigma \\ & -(d_1\alpha_4 + d_2^2\alpha_1)\|Y\|^2 + (2d_1\alpha_2 - d_2 - 4d_1^2\|\Gamma\|)\|Z\|^2 \\ & +(\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + d_1 - \frac{1}{\alpha_1})\|W\|^2.\end{aligned}$$

From (i) and lemma 2.1 we find:

$$2d_2\int_0^1 \langle \sigma\Psi(\sigma Y)Y,Y\rangle d\sigma \geq 2d_2\alpha_2\int_0^1 \langle Y,Y\rangle \sigma d\sigma = d_2\alpha_2\|Y\|^2.$$

Thus we get:

$$2V(X_t,Y_t,Z_t,W_t) \geq 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \langle \Gamma^{-1}H(X),H(X)\rangle$$
$$+2\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma - \langle \Gamma Y,Y\rangle + (d_2\alpha_2 - d_1\alpha_4 - d_2^2\alpha_1)\|Y\|^2$$
$$+(2d_1\alpha_2 - d_2 - 4d_1^2\|\Gamma\|)\|Z\|^2 + \left(\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + d_1 - \frac{1}{\alpha_1}\right)\|W\|^2.$$

It follows that:

$$2V(X_t,Y_t,Z_t,W_t) \geq V_1 + V_2 + V_3 + V_4, \qquad where$$
$$V_1 := 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \langle \Gamma^{-1}H(X),H(X)\rangle,$$
$$V_2 := 2\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma - \langle \Gamma Y,Y\rangle + (d_2\alpha_2 - d_1\alpha_4 - d_2^2\alpha_1)\|Y\|^2,$$
$$V_3 := (2d_1\alpha_2 - d_2 - 4d_1^2\|\Gamma\|)\|Z\|^2,$$
$$V_4 := \left(\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + d_1 - \frac{1}{\alpha_1}\right)\|W\|^2.$$

First to estimate V_1 we know that:

$$\frac{\partial}{\partial\sigma_1}\langle H(\sigma_1 X),H(\sigma_1 X)\rangle = 2\langle J_H(\sigma_1 X)X,H(\sigma_1 X)\rangle,$$

by integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$, and because of $H(0) = 0$, we obtain:

$$\langle H(X),H(X)\rangle = 2\int_0^1 \langle J_H(\sigma_1 X)X,H(\sigma_1 X)\rangle d\sigma_1.$$

Hence:

$$V_1 = 2d_2\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \langle \Gamma^{-1}H(X),H(X)\rangle$$
$$= 2\int_0^1 \langle H(\sigma_1 X),\{d_2 I - \Gamma^{-1}J_H(\sigma_1 X)\}X\rangle d\sigma_1.$$

But from

$$\frac{\partial}{\partial\sigma_2}\langle H(\sigma_1\sigma_2 X),\{d_2 I - \Gamma^{-1}J_H(\sigma_1 X)\}X\rangle$$
$$= \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X,\{d_2 I - \Gamma^{-1}J_H(\sigma_1 X)\}X\rangle,$$

by integrating both sides from $\sigma_2 = 0$ to $\sigma_2 = 1$, and since $H(0) = 0$, we find:

$$\langle H(\sigma_1 X),\{d_2 I - \Gamma^{-1}J_H(\sigma_1 X)\}X\rangle$$
$$= \int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2 X)X,\{d_2 I - \Gamma^{-1}J_H(\sigma_1 X)\}X\rangle d\sigma_2.$$

Hence by using (2.3), (3.4), (v), (vi) and lemma 2.1 we get:

$$\begin{aligned}
V_1 &= 2\int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2X)X, \{d_2I - \Gamma^{-1}J_H(\sigma_1X)\}X\rangle d\sigma_1 d\sigma_2 \\
&= 2\int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2X)\{d_2I - \Gamma^{-1}J_H(\sigma_1X)\}X, X\rangle d\sigma_1 d\sigma_2 \\
&\geq 2\varepsilon\int_0^1\int_0^1 \langle J_H(\sigma_1\sigma_2X)\sigma_1X, X\rangle d\sigma_1 d\sigma_2 \\
&\quad + \frac{4}{\alpha_3}\int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2X)X, \{\frac{1}{2}\alpha_4 I - J_H(\sigma_1X)\}X\rangle d\sigma_1 d\sigma_2 \\
&\geq 2\varepsilon\int_0^1\left[\int_0^1 \langle J_H(\sigma_2\tilde{X})\tilde{X}, X\rangle d\sigma_2\right] d\sigma_1 \\
&\geq 2\varepsilon\int_0^1 \alpha_4'\langle\tilde{X}, X\rangle d\sigma_1 \;\geq \varepsilon\alpha_4'\|X\|^2. \qquad (3.6)
\end{aligned}$$

Second to estimate V_2 we need:

$$\alpha_2 d_2 - \alpha_4 d_1 - \alpha_1 d_2^2 = d_2\{\alpha_2 - 4d_1\|J_G(Y)\| - \alpha_1 d_2\} + d_1\{4d_2\|J_G(Y)\| - \alpha_4\},$$

but from (2.3) and (ii) we get:

$$4d_2\big||J_G(Y)|\big| - \alpha_4 \geq 2\left(\varepsilon + \frac{\alpha_4}{\alpha_3}\right)\alpha_3 - \alpha_4 = 2\alpha_3\varepsilon + \alpha_4 > 0.$$

Hence we have:

$$\alpha_2 d_2 - \alpha_4 d_1 - \alpha_1 d_2^2 \geq d_2\{\alpha_2 - 4d_1\|J_G(Y)\| - \alpha_1 d_2\}.$$

Here, from (2.3) and (iii) we obtain:

$$\begin{aligned}
\alpha_2 - 4d_1\|J_G(Y)\| - \alpha_1 d_2 &= \alpha_2 - \frac{4}{\alpha_1}\|J_G(Y)\| - \frac{\alpha_4}{\alpha_3}\alpha_1 - \varepsilon\{4\|J_G(Y)\| + \alpha_1\} \\
&= \frac{1}{\alpha_1\alpha_3}[\{\alpha_1\alpha_2 - 4\|J_G(Y)\|\}\alpha_3 - \alpha_1^2\alpha_4] - \varepsilon\{4\|J_G(Y)\| + \alpha_1\} \\
&\geq \frac{\Delta}{\alpha_1\alpha_3} - \varepsilon(\alpha_1 + \alpha_1\alpha_2) = \frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0;\ \text{by } (iii) \quad (3.7)
\end{aligned}$$

From the identity:

$$\int_0^1 \sigma\langle J_G(\sigma Y)Y, Y\rangle d\sigma \equiv \langle G(Y), Y\rangle - \int_0^1 \langle G(\sigma Y), Y\rangle d\sigma,$$

it follows from (3.3) and by lemma 2.1:

$$\begin{aligned}
2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma - \langle G(Y), Y\rangle &= \int_0^1 \langle G(\sigma Y), Y\rangle d\sigma - \int_0^1 \sigma\langle J_G(\sigma Y)Y, Y\rangle d\sigma \\
&= -\int_0^1 \sigma\langle\{J_G(\sigma Y) - \Gamma(\sigma Y)\}Y, Y\rangle d\sigma \\
&\geq -\frac{1}{2}\delta\|Y\|^2;\ \text{by } (iv) \qquad (3.8)
\end{aligned}$$

Hence

$$V_2 \geq d_2(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0)||Y||^2 - \frac{1}{2}\delta\, \|Y\|^2$$
$$\geq \{\frac{\alpha_4}{\alpha_3}(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0) - \frac{1}{2}\delta\}\ \ \|Y\|^2$$
$$\geq \frac{1}{4}(\frac{2\alpha_4\Delta}{\alpha_1\alpha_3^2} - \delta)\|Y\|^2, \qquad since\ \varepsilon < \frac{\alpha_3}{4\alpha_4 D_0}(\frac{2\Delta\alpha_4}{\alpha_1\alpha_3^2} - \delta). \qquad (3.9)$$

Third to estimate V_3 we need:

$$2d_1\alpha_2 - d_2 - 4d_1^2\|\Gamma\| = d_1\{\alpha_2 - 4d_1\|\Gamma\| - \alpha_1 d_2\} + d_2(\alpha_1 d_1 - 1) + d_1\alpha_2$$
$$\geq d_1\{\alpha_2 - 4d_1\|J_G(Y)\| - \alpha_1 d_2\}$$
$$\geq d_1(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0);$$

by (3.5) and (3.7). Then by using (2.3) we get:

$$V_3 = (2d_1\alpha_2 - d_2 - 4d_1^2\|\Gamma\|)\|Z\|^2$$
$$\geq \frac{1}{\alpha_1}(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0)\|Z\|^2$$
$$\geq \frac{3}{4}(\frac{\Delta}{\alpha_1^2\alpha_3})\|Z\|^2, \qquad since\ \varepsilon < \frac{\Delta}{4D_0\alpha_1\alpha_3}. \qquad (3.10)$$

Finally since

$$V_4 := (\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + d_1 - \frac{1}{\alpha_1})\|W\|^2,$$

from (2.3) we get:

$$V_4 := \left(\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + \varepsilon\right)\|W\|^2. \qquad (3.11)$$

Therefore from (3.6), (3,9), (3.10) and (3.11) we obtain:

$$2V(X_t, Y_t, Z_t, W_t) \geq \varepsilon\alpha_4'\|X\|^2 + \frac{1}{4}(\frac{2\alpha_4\Delta}{\alpha_1\alpha_3^2} - \delta)\|Y\|^2 + \frac{3\Delta}{4\alpha_1^2\alpha_3}\|Z\|^2$$
$$+ \left(\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + \varepsilon\right)\|W\|^2. \qquad (3.12)$$

Since the coefficients are positive constants from (3.12), then there exists a positive constant D_1, such that:

$$V(X_t, Y_t, Z_t, W_t) \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2), \qquad (3.13)$$
$$where\ \ D_1 = \frac{1}{2}\min\left\{\varepsilon\alpha_4', \frac{1}{4}(\frac{2\alpha_4\Delta}{\alpha_1\alpha_3^2} - \delta), \frac{3\Delta}{4\alpha_1^2\alpha_3}, (\frac{d_1\alpha_1^2\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0} + \varepsilon)\right\}.$$

This derives that:

$$V(X_t, Y_t, Z_t, W_t) \geq 0, \quad \text{if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 \geq 0,$$

$$V(X_t, Y_t, Z_t, W_t) \to \infty, \quad \text{if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 \to \infty,$$

which satisfies the left hand side of the inequality in the condition (i) of Theorem 2.1.

Now we will prove the right hand side of the inequality in the condition (i) of Theorem 2.1.

$$V(X_t, Y_t, Z_t, W_t) \leq D_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2),$$

for some positive constant D_2.

By using the hypotheses of Theorem 2.2, we find:

$F(0) = 0$ and $\frac{\partial F(\sigma W)}{\partial \sigma} = J_F(\sigma W)W$, then from (vii) we get:

$$\|F(W)\| \leq \sqrt{n}\left(\alpha_1 + \frac{2\varepsilon_0\alpha_3^3}{\alpha_1\alpha_4^2}\right)\|\mathrm{W}\|, \tag{3.14}$$

from (i) we find:

$$\|\Psi(Y)\| \leq \sqrt{n}\left(\alpha_2 + \tfrac{1}{4}\alpha_1^3\varepsilon\right), \tag{3.15}$$

also since $G(0) = 0$ and $\frac{\partial G(\sigma Y)}{\partial \sigma} = J_G(\sigma Y)Y$, then from (iii) we get:

$$\|G(Y)\| \leq \tfrac{1}{4}\sqrt{n}\alpha_1\alpha_2\|Y\|, \tag{3.16}$$

since $H(0) = 0$ and $\frac{\partial H(\sigma X)}{\partial \sigma} = J_H(\sigma X)X$, then from (vi) we get:

$$\|H(X)\| \leq \tfrac{1}{2}\sqrt{n}\alpha_4\|X\|. \tag{3.17}$$

Also from

$$\begin{aligned} 2\mu\int_{-r}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds &\leq 2\mu\|Y\|^2\int_{t-r}^{t}(\theta - t + r)d\theta \\ &\leq \mu r^2\|Y\|^2, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} 2\lambda\int_{-r}^{0}\int_{t+s}^{t}\|Z(\theta)\|^2 d\theta ds &\leq 2\lambda\|Z\|^2\int_{t-r}^{t}(\theta - t + r)d\theta \\ &\leq \lambda r^2\|Z\|^2. \end{aligned} \tag{3.19}$$

Hence by using Cauchy-Schwarz inequality $|\langle U, V\rangle| \leq \frac{1}{2}(\|U\|^2 + \|V\|^2)$ and from (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) we get:

$$\begin{aligned} 2V(X_t, Y_t, Z_t, W_t) \leq &\left\{\frac{1}{12}nd_2\alpha_4^2 + d_2 + \frac{1}{4}n\alpha_4^2(2d_1 + 1)\right\}\|X\|^2 \\ &+\left\{\frac{1}{16}n\alpha_1^2\alpha_2^2(2d_1 + \frac{1}{3}) + d_2\left[n(\alpha_2 + \frac{1}{4}\alpha_1^3\varepsilon)^2 + \alpha_1 + 2\right] + \mu r^2 + 2\right\}\|Y\|^2 \\ &+\{2d_1(\alpha_2 + 2) + \alpha_1(d_2 + 1) + \lambda r^2 + 1\}\|Z\|^2 \\ &+\left\{d_1\left[\frac{1}{3}n\left(\alpha_1 + \frac{2\varepsilon_0\alpha_3^3}{\alpha_1\alpha_4^2}\right) + \left(\frac{\alpha_1\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0}\right) + 1\right] + d_2 + 1\right\}\|W\|^2. \end{aligned} \tag{3.20}$$

Hence there exists a positive constant D_2 satisfying:

$$V(X_t, Y_t, Z_t, W_t) \leq D_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2). \tag{3.21}$$

This completes the right hand side of the inequality in the condition (i)

of Theorem2.1.

Now we prove that $\dot{V}(X_t, Y_t, Z_t, W_t) \leq 0$, from (3.1), (3.2) and lemma 2.2, we get:

$$\frac{dV}{dt}(X_t, Y_t, Z_t, W_t) = d_2\langle H(X), Y\rangle - d_1\langle \alpha_4 Y, Z\rangle + \langle G(Y), Z\rangle + d_2\langle \Psi(Y)Z, Y\rangle$$
$$+2d_1\langle \alpha_2 Z, W\rangle + (\alpha_1 - d_2)\langle Z, W\rangle + d_2\langle \alpha_1 Z, Z\rangle + d_2\langle \alpha_1 W, Y\rangle$$
$$+d_1\left(\frac{\alpha_1\alpha_4^2}{\alpha_1^2\alpha_4^2 + 2\alpha_3^3\varepsilon_0}\right)\langle F(W), \dot{W}\rangle + d_1\langle W, \dot{W}\rangle + \langle J_H(X)Y, Y\rangle$$
$$+\langle H(X), Z\rangle + 2d_1\langle J_H(X)Y, Z\rangle + 2d_1\langle H(X), W\rangle + 2d_1\langle J_G(Y)Z, Z\rangle$$
$$+2d_1\langle G(Y), W\rangle + d_2\langle Y, \dot{W}\rangle + d_2\langle Z, W\rangle + \langle W, W\rangle + \langle Z, \dot{W}\rangle$$
$$+\mu r\|Y\|^2 - \mu\int_{t-r}^{t}\|Y(\theta)\|^2 d\theta + \lambda r\|Z\|^2 - \lambda\int_{t-r}^{t}\|Z(\theta)\|^2\, d\theta.$$

By simple calculations, we obtain:

$$\frac{dV}{dt} \leq -\{d_2\langle Y, G(Y)\rangle - \frac{1}{2}\alpha_4\|Y\|^2\} - (\alpha_2 - 2d_1\|J_G(Y)\| - d_2\alpha_1)\|Z\|^2$$
$$-(2\alpha_1 d_1 - 1)\|W\|^2 + \langle 2d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_G(Y(s))Z(s)ds\rangle$$
$$+\langle 2d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_H\big(X(s)\big)Y(s)ds\rangle + \mu\|Y\|^2 r + \lambda\|Z\|^2 r$$
$$-\mu\int_{t-r}^{t}\|Y(\theta)\|^2 d\theta - \lambda\int_{t-r}^{t}\|Z(\theta)\|^2\, d\theta + V_5 + V_6, \qquad (3.22)$$

where

$V_5 := -2d_1\langle W, F(W)\rangle + 2d_1\langle \alpha_1 W, W\rangle - d_2\langle Y, F(W)\rangle + d_2\langle \alpha_1 W, Y\rangle.$

$V_6 := -\langle Z, \Psi(Y)Z\rangle + \langle \alpha_2 Z, Z\rangle - 2d_1\langle W, \Psi(Y)Z\rangle + 2d_1\langle \alpha_2 Z, W\rangle.$

But :

$$V_5 = -2d_1\int_0^1\left[\langle J_F(\sigma W)W, W\rangle - \langle \alpha_1 W, W\rangle + \frac{d_2}{2d_1}\{\langle J_F(\sigma W)W, Y\rangle - \langle \alpha_1 W, Y\rangle\}\right]d\sigma$$
$$= -2d_1\left[\int_0^1\langle\{J_F(\sigma W) - \alpha_1 I\}W, W\rangle d\sigma + \frac{d_2}{2d_1}\int_0^1\langle\{J_F(\sigma W) - \alpha_1 I\}W, Y\rangle d\sigma\right].$$

Since $\lambda_i(\int_0^1 J_F(\sigma W)d\sigma - \alpha_1 I)$ is non-negative by (vii), then from (2.3) we get:

$$V_5 \leq \frac{d_2^2}{8d_1}\int_0^1\langle\{J_F(\sigma W) - \alpha_1 I\}Y, Y\rangle d\sigma$$
$$\leq \frac{(\alpha_3\alpha_4^{-1}\varepsilon + 1)^2}{4(\alpha_1\varepsilon + 1)}\varepsilon_0\alpha_3\|Y\|^2 \leq \frac{1}{2}\varepsilon_0\alpha_3\|Y\|^2, \qquad \text{because of } \varepsilon < \left\{\frac{1}{\alpha_1}, \frac{\alpha_4}{\alpha_3}\right\}.$$

Also

$$\begin{aligned} V_6 &= -[\langle Z, \Psi(Y)Z\rangle - \langle \alpha_2 Z, Z\rangle + 2d_1\{\langle W, \Psi(Y)Z\rangle - \langle \alpha_2 Z, W\rangle\}] \\ &= -[\langle\{\Psi(Y) - \alpha_2 I\}Z, Z\rangle + 2d_1\langle\{\Psi(Y) - \alpha_2 I\}Z, W\rangle], \end{aligned}$$

since $(\Psi(Y) - \alpha_2 I)$ is non-negative by (i), then from (2.3) we get :

$$\begin{aligned} V_6 &\le d_1^2\langle\{\Psi(Y) - \alpha_2 I\}W, W\rangle \\ &\le \tfrac{1}{4}(\alpha_1\varepsilon + 1)^2\alpha_1\varepsilon\|W\|^2 \le \alpha_1\varepsilon\|W\|^2, \text{ because of } \varepsilon < \tfrac{1}{\alpha_1}. \end{aligned}$$

Therefore from (3.23) and (3.24) we get:

$$\begin{aligned} \frac{dV}{dt} &\le -\frac{1}{2}\{d_2\alpha_3 - \alpha_4 - \varepsilon_0\alpha_3\}\|Y\|^2 - \{\alpha_2 - 2d_1\|J_G(Y)\| - d_2\alpha_1\}\|Z\|^2 \\ &\quad -(2\alpha_1 d_1 - \alpha_1\varepsilon - 1)\|W\|^2 + \langle 2d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_H(X(s))Y(s)ds\rangle \\ &\quad +\langle 2d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_G(Y(s))Z(s)ds\rangle + \mu\|Y\|^2 r + \lambda\|Z\|^2 r \\ &\quad -\mu\int_{t-r}^{t}\|Y(\theta)\|^2\, d\theta - \lambda\int_{t-r}^{t}\|Z(\theta)\|^2\, d\theta, \end{aligned}$$

since $-d_2\langle Y, G(Y)\rangle = -d_2\langle Y, \Gamma(Y)Y\rangle \le -\frac{1}{2}d_2\alpha_3\langle Y, Y\rangle$ from (3.4).

Here, since $\|J_H(X)\| \le \frac{1}{2}\alpha_4\sqrt{n}$ by (vi) and by using Cauchy-Schwarz inequality, we find:

$$\begin{aligned} &\left|\langle 2d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_H(X(s))Y(s)ds\rangle\right| \\ &\qquad \le \|2d_1 W + Z + d_2 Y\|\left\|\int_{t-r}^{t} J_H(X(s))Y(s)ds\right\| \\ &\qquad \le \frac{d_1\alpha_4\sqrt{n}}{2}(\|W\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2\, ds) \\ &\qquad\quad + \frac{\alpha_4\sqrt{n}}{4}(\|Z\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2 ds) \\ &\qquad\quad + \frac{d_2\alpha_4\sqrt{n}}{4}(\|Y\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2\, ds). \end{aligned}$$

Also since $\|J_G(Y)\| \le \frac{1}{4}\alpha_1\alpha_2\sqrt{n}$ by (iii) and by using Cauchy-Schwarz inequality, we find:

$$\left|\left\langle 2d_1W + Z + d_2Y, \int_{t-r}^{t} J_G(Y(s))Z(s)ds\right\rangle\right|$$

$$\leq \|2d_1W + Z + d_2Y\| \left\| \int_{t-r}^{t} J_G(Y(s))Z(s)ds \right\|$$

$$\leq \frac{d_1\alpha_1\alpha_2\sqrt{n}}{4}(\|W\|^2 r + \int_{t-r}^{t} \|Z(s)\|^2\, ds)$$

$$+\frac{\alpha_1\alpha_2\sqrt{n}}{8}(\|\mathrm{Z}\|^2 r + \int_{tr}^{t} \|Z(s)\|^2 ds)$$

$$+\frac{d_2\alpha_1\alpha_2\sqrt{n}}{8}(\|Y\|^2 r + \int_{t-r}^{t} \|Z(s)\|^2\, ds).$$

Therefore it follows from (2.2), (2.3) and (3.7) that:

$$\frac{dV}{dt} \leq -\left\{\frac{1}{2}(\varepsilon - \varepsilon_0)\alpha_3 - \frac{d_2\alpha_4\sqrt{n}}{4}r - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{8}r - \mu r\right\}\|\mathrm{Y}\|^2$$

$$-\left\{(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0) - \frac{\alpha_4\sqrt{n}}{4}r - \frac{\alpha_1\alpha_2\sqrt{n}}{8}r - \lambda r\right\}\|\mathrm{Z}\|^2$$

$$-\left(\alpha_1\varepsilon + 1 - \frac{d_1\alpha_4\sqrt{n}}{2}r - \frac{d_1\alpha_1\alpha_2\sqrt{n}}{4}r\right)\|\mathrm{W}\|^2$$

$$+\left(\frac{2d_1\alpha_4\sqrt{n}}{4} + \frac{d_2\alpha_4\sqrt{n}}{4} + \frac{\alpha_4\sqrt{n}}{4} - \mu\right)\int_{t-r}^{t} \|\mathrm{Y(s)}\|^2 ds$$

$$+\left(\frac{2d_1\alpha_1\alpha_2\sqrt{n}}{8} + \frac{d_2\alpha_1\alpha_2\sqrt{n}}{8} + \frac{\alpha_1\alpha_2\sqrt{n}}{8} - \lambda\right)\int_{t-r}^{t} \|\mathrm{Z(s)}\|^2 ds.$$

If we take

$$\mu = \frac{\alpha_4\sqrt{n}}{4}(2d_1 + d_2 + 1) \text{ and } \lambda = \frac{\alpha_1\alpha_2\sqrt{n}}{8}(2d_1 + d_2 + 1),$$

then we have:

$$\frac{dV}{dt} \leq -\left\{\frac{1}{2}(\varepsilon - \varepsilon_0)\alpha_3 - \frac{d_2\alpha_4\sqrt{n}}{4}r - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{8}r - \frac{\alpha_4\sqrt{n}}{4}(2d_1 + d_2 + 1)r\right\}\|\mathrm{Y}\|^2$$

$$-\left\{(\frac{\Delta}{\alpha_1\alpha_3} - \varepsilon D_0) - \frac{\alpha_4\sqrt{n}}{4}r - \frac{\alpha_1\alpha_2\sqrt{n}}{8}r - \frac{\alpha_1\alpha_2\sqrt{n}}{8}(2d_1 + d_2 + 1)r\right\}\|\mathrm{Z}\|^2$$

$$-\left\{(\alpha_1\varepsilon + 1) - \frac{d_1\alpha_4\sqrt{n}}{2}r - \frac{d_1\alpha_1\alpha_2\sqrt{n}}{4}r\right\}\|\mathrm{W}\|^2.$$

Therefore if

$$r < \min\left\{\frac{4(\varepsilon-\varepsilon_0)\alpha_3}{2\alpha_4\sqrt{n}(2d_1+2d_2+1)+\alpha_1\alpha_2 d_2\sqrt{n}},\right.$$
$$\left.\frac{8(\frac{\Delta}{\alpha_1\alpha_3}-\varepsilon D_0)}{\sqrt{n}\{2\alpha_4+\alpha_1\alpha_2(2d_1+d_2+2)\}},\frac{4(\alpha_1\varepsilon+1)}{d_1\sqrt{n}(\alpha_1\alpha_2+2\alpha_4)}\right\}$$

we obtain:

$$\frac{dV}{dt} \leq -D_3(\|Y\|^2+\|Z\|^2+\|W\|^2), \qquad \text{for some } D_3>0. \qquad (3.26)$$

Therefore from (3.13), (3.21) and (3.26) the functional $V(X_t, Y_t, Z_t, W_t)$ satisfies all conditions of the Theorem 2.1, so that the zero solution of (1.1) is uniformaly stable. Thus the proof of Theorem 2.2 is completed.

4 Illustrative Example

We display an example to illustrate the sufficient conditions which given in Theorem 2.2.

Example : In a special case of equation (1.1), for n = 2; we choose

$$F(W)=\begin{pmatrix} w_1(t)+w_1^3(t) \\ w_2(t)+w_2^3(t)\end{pmatrix}, \qquad \Psi(Y)=\begin{pmatrix} 5+y_1^2(t) & 0 \\ 0 & 5+y_1^2(t)\end{pmatrix},$$

$$G(Y(t-r))=\begin{pmatrix} y_1(t-r) \\ y_2(t-r)\end{pmatrix},$$

$$H(X(t-r))=\begin{pmatrix}\frac{1}{4}x_1(t-r)+\frac{1}{2}\arctan(x_1(t-r)) \\ \frac{1}{4}x_2(t-r)+\frac{1}{2}\arctan(x_2(t-r)\end{pmatrix}.$$

We find that $F(0)=0$ and

$$J_F(W)=\begin{pmatrix} 1+3w_1^2(t) & 0 \\ 0 & 1+3w_2^2(t)\end{pmatrix},$$

is symmetric, and

$$\int_0^1 J_F(\sigma W)d\sigma=\begin{pmatrix} 1+w_1^2(t) & 0 \\ 0 & 1+w_2^2(t)\end{pmatrix},$$

so, we obtain:

$$\lambda_1(\int_0^1 J_F(\sigma W)d\sigma)=1+w_1^2(t), \ \lambda_2(\int_0^1 J_F(\sigma W)d\sigma)=1+w_2^2(t),$$

therefore we get $\lambda_i(\int_0^1 J_F(\sigma W)d\sigma) \geq 1,\ \alpha_1 = 1.$

Also, we can see that the matrix $\Psi(Y)$ is symmetric, and

$$\lambda_1(\Psi(Y)) = 5 + y_1^2(t),\ \ \lambda_2(\Psi(Y)) = 5 + y_2^2(t),$$

then, we have $\lambda_i(\Psi(Y)) \geq 5,\ \alpha_2 = 5.$
Also, we find that $G(0) = 0$ and

$$J_G(Y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is symmetric, and

$$\int_0^1 J_G(\sigma Y)d\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then, we get:

$$\lambda_1(\int_0^1 J_G(\sigma Y)d\sigma) = 1,\ \ \lambda_2(\int_0^1 J_G(\sigma Y)d\sigma) = 1,$$

therefore we obtain $\lambda_i(\int_0^1 J_G(\sigma Y)) = 1,\ \alpha_3 = 2.$

Likewise, we have $H(0) = 0,$

$$J_H(X) = \begin{pmatrix} \frac{1}{4} + \frac{1}{2(1 + x_1^2(t-r))} & 0 \\ 0 & \frac{1}{4} + \frac{1}{2(1 + x_2^2(t-r))} \end{pmatrix},$$

is symmetric, and

$$\int_0^1 J_H(\sigma X)d\sigma = \begin{pmatrix} \frac{1}{4} + \frac{1}{2x_1(t-r)}\arctan x_1(t-r) & 0 \\ 0 & \frac{1}{4} + \frac{1}{2x_2(t-r)}\arctan x_2(t-r) \end{pmatrix},$$

then, we obtain:

$$\lambda_1(\int_0^1 J_H(\sigma X)d\sigma) = \frac{1}{4} + \frac{1}{2x_1(t-r)}\arctan x_1(t-r),$$

$$\lambda_2(\int_0^1 J_H(\sigma X)d\sigma) = \frac{1}{4} + \frac{1}{2x_2(t-r)}\arctan x_2(t-r),$$

therefore we get $\lambda_i\left(\int_0^1 J_H(\sigma X)\right) \geq \frac{1}{4},\ \ \alpha_4' = \frac{1}{4}.$

$J_H(X)$ commutes with $J_H(X')$, for all $X, X' \in \mathcal{R}^n$ and

$$\lambda_1(J_H(X)) = \frac{1}{4} + \frac{1}{2(1 + x_1^2(t-r))}, \;\; \lambda_2(J_H(X)) = \frac{1}{4} + \frac{1}{2(1 + x_2^2(t-r))},$$

therefore we get $\lambda_i\big(J_H(X)\big) \leq \frac{3}{4}, \;\; \alpha_4 = \frac{3}{2}.$

Now, since $\|J_G(Y)\| = \sqrt{\lambda_{max}\big(J_G^*(Y)J_G(Y)\big)} = 1,$ where $J_G^*(Y)$ is transpose of matrix $J_G(Y)$, then there is a finite positive constant

$$\Delta \leq \{\alpha_1\alpha_2 - 4\|J_G(Y)\|\}\alpha_3 - \alpha_1^2\alpha_4 = \frac{1}{2}.$$

Finally, we have $0 \leq \lambda_i\left(J_G(Y) - \int_0^1 J_G(\sigma Y)\right) \leq \delta < \frac{3}{8},$

and choose $\delta = \frac{1}{4}$, we get:

$$\varepsilon_0 < \varepsilon = \min\{1, \frac{3}{4}, \frac{1}{96}, \frac{1}{144}\} \approx 0.0069444444.$$

If we take $\varepsilon_0 = 0.006$, then all conditions of Theorem 2.2 are hold provided that:

$$r < \min\{0.0003076098, 0.0438856032, 0.3535533906\} \approx 0.0003076098.$$

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دراسة استقرار الحلول لمعادلة تفاضلية تأخيرية اتجاهية معينة من الرتبة الرابعة

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في هذا البحث، باستخدام دالية ليابونوف نناقش الشروط الكافية لدراسة الاستقرار المنتظم للحل الصفري لمعادلة تفاضلية غير خطية تأخيرية اتجاهية من الرتبة الرابعة على الصورة:

$$X^{(4)} + \mathrm{F}(\dddot{X}) + \Psi(\dot{X})\,\ddot{X} + \mathrm{G}\left(\dot{X}(\mathrm{t}-\mathrm{r})\right) + \mathrm{H}\big(\mathrm{X}(\mathrm{t}-\mathrm{r})\big) = 0.$$

النتائج التي تم الحصول عليها في هذا البحث هي تحسين لبعض النتائج التي تم دراستها مسبقا، مع ذكر مثال يوضح النتيجة الرئيسية التي اسفر عنها البحث.