Assiut Univ. J. of Mathematics and Computer Science *Printed ISSN 1687-4897*
48(2), pp.125-139(2019)

ESTIMATION IN CONSTANT-STRESS PARTIALLY ACCELERATED LIFE TESTS FOR BURR TYPE XII USING TAMPERED RANDOM VARIABLE MODEL UNDER UNIFIED HYBRID CENSORING DATA

Ayman M. Abd-Elrahman, Abd EL-Baset A. Ahmad and Marwa A. Younis [1]
Department of Mathematics, Faculty of Science,
Assuit University, Assuit 71515, Egypt.

Received: 12/9/2019 **Accepted:** 7/11/2019 **Available Online:** 1/12/2019

This paper discusses constant stress partially accelerated life tests for unified hybrid censoring data from Burr type XII distribution with a tampered random variable. Both maximum likelihood and Bayesian methods are used to estimate the unknown population parameters and accelerated factor. In order to compute the asymptotic confidence intervals for the maximum likelihood estimators, we first calculate the Fisher information matrix related to the underlying model. On the other hand, Markov Chain Monte Carlo method under squared error loss function is used for obtaining the corresponding Bayesian estimators. In addition, a simulation study is carried out to compare the performances of the resulting estimators.

Keywords: Unified hybrid censoring; Maximum likelihood estimators; Bayesian estimators; Observed Fisher information matrix; Markov Chain Monte Carlo method.

1 Introduction

The technological development in all aspects of life, especially reliability tests, led to the rapid desire to know the reliability of the product, unit or system in order to improve quality, reduce costs and try to satisfy customers. The most popular way to do this, is to adopt accelerated life tests (ALTs) and partially accelerated life tests (PALTs). These tests can be carried out with different stress loading such as constant stress (CS), step stress and others. As for the observed data, it may be complete data or censoring data. Censored data such as, Type I, Type II, progressive, hybrid, unified hybrid, etc. To know more about life tests, types of stress loading and data types, see Nelson's book [9].

In this study, we will adopt CS-PALTs, where the test units exposed to

[1] Corresponding author e-mail: marwa.younes@science.au.edu.eg

both normal and accelerated conditions. In CS, the units are divided into two groups, one of them is being accelerated and the other under normal conditions. Here, following DeGroot et al. [6], we use data come from a tampered random variable (TRV) model, where the acceleration factor is the ratio of mean lifetime at use conditions to that at accelerated conditions. There are many studies for CS-PALTs, among them, Abdel-Hamid [1], Cheng et al. [5] and Ahmad et al. [2]. Also, we will adopt unified hybrid censoring scheme (U-HCS) which proposed by Balakrishnan et al. [3], U-HCS is a mixture of generalized hybrid Type-I and generalized hybrid Type-II. Suppose n identical units are put on life test. In U-HCS, one fixes $k, r \in \{1,2,\dots,n\}$ and $T_1, T_2 \in (0,\infty)$ such that $k < r < n$ and $T_1 < T_2$. If k^{th} failures occurs before T_1 the test terminate at $min\{max\{Y_{r:n}, T_1\}, T_2\}$, if k^{th} failures occurs between T_1 and T_2 the test terminate at $min\{Y_{r:n}, T_2\}$ and if k^{th} failures occurs after T_2 the test terminate at $Y_{k:n}$.

Based on such a U-HCS, the experiment will be completed at time C with D ordered observations. Here, $C \in \{Y_{k:n}, Y_{r:n}, T_1, T_2\}$, which is the terminated time of the experiment and $D \in \{k, r, d_1, d_2\}$ is the number of failure units until the time C with $d_i, i = 1,2$, is the number of failure until the time T_i. Therefor, under the U-HCS, described above, we have six cases as shown in Table 1 We can summarize the U-HCS six cases according to Table 1. U-HCS is discussed by many authors, for example, Habibi Rad et al. [7], [8] and Panahi et al. [10], [11].

Table 1: Summary of cases in which an experiment under U-HCS will be completed.

Case		Terminated time C	Number of failure units D
I	$0 < Y_{k:n} < Y_{r:n} < T_1 < T_2$	T_1	d_1
II	$0 < Y_{k:n} < T_1 < Y_{r:n} < T_2$	$Y_{r:n}$	r
III	$0 < Y_{k:n} < T_1 < T_2 < Y_{r:n}$	T_2	d_2
IV	$0 < T_1 < Y_{k:n} < Y_{r:n} < T_2$	$Y_{r:n}$	r
V	$0 < T_1 < Y_{k:n} < T_2 < Y_{r:n}$	T_2	d_2
VI	$0 < T_1 < T_2 < Y_{k:n} < Y_{r:n}$	$Y_{k:n}$	k

In this article, Burr type XII, $Burr(\alpha, \beta)$, is a two-parameter distribution that was introduced by Burr [4]. It can be applied in many areas as a life time model for fitting data that comes from various practical fields, e. g., biological, clinical, or many experimental data. The following section presents a brief description for the underlying model.

2 Model Description

In this article, we assumed that the lifetimes of units under the test, which come from a TRV model, are independent random variables follow $Burr(\alpha,\beta)$. The n units which constitute the complete sample from the underlying population is divided into two groups with sizes n_1 and n_2. The test begins at time zero, and U-HCS is observed for each group, where the first group has been treated under the normal or usual condition, which is called use conditions. While the second group has been treated under stress conditions. We can write this as Group 1 where $n_1 = n - n\pi$ units chosen randomly in normal use conditions (π is the proportion of sample units allocated to accelerated conditions), and Group 2 where $n_2 = n\pi$ units remain in accelerated conditions.

Let the lifetime Y of an item tested for Group 1 at use conditions follows $Burr(\alpha,\beta)$ with CDF and PDF are given by

$$F_1(y;\alpha,\beta) = 1 - (1 + y^\alpha)^{-\beta}, y > 0, (\alpha,\beta > 0),$$
$$f_1(y;\alpha,\beta) = \alpha\, \beta\, y^{\alpha-1}(1 + y^\alpha)^{-(\beta+1)},$$

and let the lifetime Y of an item tested for Group 2 at accelerated conditions with CDF and PDF are given by

$$F_2(y;\alpha,\beta,\theta) = 1 - [1 + (\theta y)^\alpha]^{-\beta}, \theta > 1,$$
$$f_2(y;\alpha,\beta,\theta) = \alpha\, \beta\, \theta^\alpha\, y^{\alpha-1}[1 + (\theta y)^\alpha]^{-(\beta+1)}.$$

Let $Y_{ji}, j = 1,2, i = 1,\dots,n_j$ are the lifetimes for the tested items allocated from Burr type XII, where $Y_{1i}, i = 1,\dots,n_1$ is the lifetime in use condition and $Y_{2i}, i = 1,\dots,n_2$ is the lifetime in accelerated condition. Then we can apply U-HCS for Group j, $j = 1,2$ as shown in Table 2.

Table 2: Cases of Group j, $j = 1,2$, in which an experiment under U-HCS will be completed.

Case		Terminated time C_j	Number of failure units D_j
I	$0 < Y_{jk_j} < Y_{jr_j} < T_{j_1} < T_{j_2}$	T_{j_1}	d_{j_1}
II	$0 < Y_{jk_j} < T_{j_1} < Y_{jr_j} < T_{j_2}$	Y_{jr_j}	r_j
III	$0 < Y_{jk_j} < T_{j_1} < T_{j_2} < Y_{jr_j}$	T_{j_2}	d_{j_2}
IV	$0 < T_{j_1} < Y_{jk_j} < Y_{jr_j} < T_{j_2}$	Y_{jr_j}	r_j
V	$0 < T_{j_1} < Y_{jk_j} < T_{j_2} < Y_{jr_j}$	T_{j_2}	d_{j_2}
VI	$0 < T_{j_1} < T_{j_2} < Y_{jk_j} < Y_{jr_j}$	Y_{jk_j}	k_j

3 Maximum Likelihood Estimation

The likelihood function (LF) based on a given U-HCS can be written as

$$L(\varphi|y) = \frac{n!}{(n-D)!} \prod_{i=1}^{D} f(y_{i:n})[1-F(C)]^{n-D},$$

where φ is a vector of parameters. By applying CS-PALT for observed lifetimes taken from Burr type XII under U-HCS we get LF for Group 1 and Group 2 as:

$$L_1(\alpha,\beta|y) = N_1 \prod_{i=1}^{D_1} f_1(y_{1i})[1-F_1(C_1)]^{n_1-D_1},$$

$$L_2(\alpha,\beta,\theta|y) = N_2 \prod_{i=1}^{D_2} f_2(y_{2i})[1-F_2(C_2)]^{n_2-D_2},$$

where

$$N_1 = \frac{n_1!}{(n_1-D_1)!}, \qquad N_2 = \frac{n_2!}{(n_2-D_2)!},$$

$$\{(C_1,D_1),(C_2,D_2)\} = \begin{cases} \{(T_{11},d_{11}),(T_{21},d_{21})\} & \text{forCaseI}, \\ \{(y_{1r_1},r_1),(y_{2r_2},r_2)\} & \text{forCaseIIorIV}, \\ \{(T_{12},d_{12}),(T_{22},d_{22})\} & \text{forCaseIIIorV}, \\ \{(y_{1k_1},k_1),(y_{2k_2},k_2)\} & \text{forCaseVI}. \end{cases}$$

Then, the LF for CS-PALT under U-HCS is given by

$$\begin{aligned} L(\alpha,\beta,\theta|y) &= L_1 \times L_2, \\ &= \prod_{j=1}^{2} N_j \prod_{i=1}^{D_j} f_j(y_{ji})[1-F_j(C_j)]^{n_j-D_j}, \\ &= \prod_{j=1}^{2} N_j \prod_{i=1}^{D_j} \alpha\,\beta\,\theta^{\alpha(j-1)} {y_{ji}}^{\alpha-1}\left[1+(\theta^{j-1}y_{ji})^{\alpha}\right]^{-(\beta+1)} \\ &\quad \times \left[1+(\theta^{j-1}C_j)^{\alpha}\right]^{-\beta(n_j-D_j)}. \end{aligned} \tag{8}$$

3.1 Point estimation

By taking the first partial derivatives of the natural logarithm (ℓ) of (8) with respect to α,β and θ, we obtain:

$$\frac{\partial \ell}{\partial \alpha} = \frac{D_1 + D_2}{\alpha} + D_2 ln\theta - \sum_{j=1}^{2} \left\{ \frac{\beta\left(n_j - D_j\right)\left(\theta^{j-1}C_j\right)^{\alpha} ln\left(\theta^{j-1}C_j\right)\right)}{1 + (\theta^{j-1}C_j)^{\alpha}} \right.$$
$$\left. + \sum_{i=1}^{D_j} \frac{(j-1)(\beta+1)(\theta y_{ji})^{\alpha}\ln\theta - \left[1 - \beta(\theta^{j-1}y_{ji})^{\alpha}\right]\ln y_{ji}}{1 + (\theta^{j-1}y_{ji})^{\alpha}} \right\}, \tag{9}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{D_1 + D_2}{\beta} - \sum_{j=1}^{2} \left\{ \ln\left[1 + (\theta^{j-1}C_j)^{\alpha}\right]^{(n_j - D_j)} + \sum_{i=1}^{D_j} \ln\left[1 + (\theta^{j-1}y_{ji})^{\alpha}\right] \right\}, \tag{10}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{\alpha D_2}{\theta} - \frac{\alpha}{\theta} \left\{ \frac{\beta(n_2 - D_2)(\theta C_2)^{\alpha}}{1 + (\theta C_2)^{\alpha}} + \sum_{i=1}^{D_2} \frac{(1+\beta)(\theta y_i)^{\alpha}}{1 + (\theta y_i)^{\alpha}} \right\}. \tag{11}$$

Equating (9), (10) and (11) to zero, we obtain the maximum likelihood estimators (MLEs) for the population parameters α, β and acceleration factor θ. Clearly the solution of these equations cannot be obtained in a simple closed-form. However, some numerical techniques, e. g. Newton–Raphson method, can be used for this propose. We have used for this method the computer program "Wolfram Mathematica 10.0".

3.2 Approximate confidence intervals

The observed Fisher information matrix for the MLEs of the unknown parameters α, β and θ is given by

$$F^{-1} = \left.\begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha\, \partial \beta} & -\frac{\partial^2 \ell}{\partial \alpha\, \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \beta\, \partial \alpha} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta\, \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \theta\, \partial \alpha} & -\frac{\partial^2 \ell}{\partial \theta\, \partial \beta} & \frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix}^{-1} \right|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}}$$
$$= \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\theta}) \\ cov(\hat{\alpha},\hat{\beta}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\theta}) \\ cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\beta},\hat{\theta}) & var(\hat{\theta}) \end{pmatrix},$$

where the elements of Fisher information matrix are given by

$$-\frac{\partial^2 \ell}{\partial \alpha^2}=\frac{D_1+D_2}{\alpha^2}+\sum_{j=1}^{2}\left\{\frac{\beta\,(n_j-D_j)(\theta^{j-1}C_j)^{\alpha}\left[\ln(\theta^{j-1}C_j)\right]^2}{[1+(\theta^{j-1}C_j)^{\alpha}]^2}\right.$$

$$\left.+\sum_{i=1}^{D_j}\frac{(\beta+1)(\theta^{j-1}y_{ji})^{\alpha}\left[ln(\theta^{j-1}y_{ji})\right]^2}{[1+(\theta^{j-1}y_{ji})^{\alpha}]^2}\right\}$$

$$-\frac{\partial^2 \ell}{\partial \alpha\,\partial \beta}=\sum_{j=1}^{2}\left\{\frac{(n_j-D_j)(\theta^{j-1}C_j)^{\alpha}\ln(\theta^{j-1}C_j)}{1+(\theta^{j-1}C_j)^{\alpha}}+\sum_{i=1}^{D_j}\frac{(\theta^{j-1}y_{ji})^{\alpha}ln(\theta^{j-1}y_{ji})}{1+(\theta^{j-1}y_{ji})^{\alpha}}\right\},$$

$$-\frac{\partial^2 \ell}{\partial \alpha\,\partial \theta}=-\frac{D_2}{\theta}-\frac{1}{\theta}\left\{\frac{\beta(n_2-D_2)\,[1+(\theta C_2)^{\alpha}+\ln(\theta C_2)^{\alpha}]}{[1+(\theta C_2)^{\alpha}]^2}\right.$$
$$\left.-\sum_{i=1}^{D_2}\frac{(\beta+1)[1+(\theta y_{2i})^{\alpha}+ln(\theta y_{2i})^{\alpha}]}{[1+(\theta y_{2i})^{\alpha}]^2}\right\},$$

$$-\frac{\partial^2 \ell}{\partial \beta^2}=\frac{D_1+D_2}{\beta^2},$$

$$-\frac{\partial^2 \ell}{\partial \beta\,\partial \theta}=\frac{\alpha}{\theta}\left\{\frac{(n_2-D_2)(\theta C_2)^{\alpha}}{1+(\theta C_2)^{\alpha}}+\sum_{i=1}^{D_2}\frac{(\theta y_{2i})^{\alpha}}{1+(\theta y_{2i})^{\alpha}}\right\},$$

$$-\frac{\partial^2 \ell}{\partial \theta^2}=\frac{\alpha D_2}{\theta^2}+\frac{\alpha}{\theta^2}\left\{\frac{\beta(n_2-D_2)(\theta C_2)^{\alpha}[\alpha-1-(\theta C_2)^{\alpha}]}{[1+(\theta C_2)^{\alpha}]^2}\right.$$
$$\left.-\sum_{i=1}^{D_2}\frac{(\beta+1)(\theta y_{2i})^{\alpha}[\alpha-1-(\theta y_{2i})^{\alpha}]}{[1+(\theta y_{2i})^{\alpha}]^2}\right\}.$$

Then, the asymptotic $100\,(1-\nu)\%$ confidence interval for α,β and θ are, respectively, given by

$$\left(\hat{\alpha}\mp Z_{\nu/2}\sqrt{var(\hat{\alpha})}\right),\left(\hat{\beta}\mp Z_{\nu/2}\sqrt{var(\hat{\beta})}\right)\text{ and }\left(\hat{\theta}\mp Z_{\nu/2}\sqrt{var(\hat{\theta})}\right).$$

3 Bayesian Estimation

In this section, the Bayesian estimates of $Burr(\alpha,\beta)$ and accelerated factor θ have been obtained based on U-HCS. We use Gamma prior density for $Burr(\alpha,\beta)$, and we use the non-informative prior for θ. Therefor, the joint prior for α,β and θ is given by

$$\pi(\alpha,\beta,\theta) = \pi_1(\alpha)\,\pi_2(\beta)\,\pi_3(\theta),$$
$$\propto \alpha^{a_1-1}\beta^{a_2-1}\theta^{-1}\theta^{-1}e^{-(b_1\alpha+b_2\beta)},$$

where

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, (a_1, b_1 > 0),$$
$$\pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta}, (a_2, b_2 > 0),$$
$$\pi_3(\theta) \propto \theta^{-1}.$$

These priors are communally used in similar situations. See e.g. Soliman et al. [12] and Ahmed et al. [2]. Combining (8) and (12), the joint posterior density function of α,β and θ is given by

$$\pi^*(\alpha,\beta,\theta|\mathbf{y}) \propto \alpha^{a_1+D_1+D_2-1}\,\beta^{a_2+D_1+D_2-1}\,\theta^{\alpha D_2-1}\,e^{-(b_1\alpha+b_2\beta)}$$
$$\times \prod_{j=1}^{2}\prod_{i=1}^{D_j} y_{ji}^{(\alpha-1)}[1+(\theta^{j-1}y_{ji})^{\alpha}]^{-(\beta+1)}\left[1+(\theta^{j-1}C_j)^{\alpha}\right]^{-\beta(n_j-D_j)}. \quad (13)$$

Under SEL function, the Bayesian estimate of any function of α,β and θ, say $\psi = \psi(\alpha,\beta,\theta)$ is given by

$$\hat{\psi}_{BS} = IE(\psi|\mathbf{y}) = \int_{\theta}\int_{\beta}\int_{\alpha}\psi\,\pi^*(\alpha,\beta,\theta|\mathbf{y})\,d\alpha\,d\beta\,d\theta\,.$$

Equation (14) cannot be obtained in a simple closed form. So, we consider Markov Chain Monte Carlo (MCMC) method to obtain Bayesian estimates for α,β and θ based on the Metropolis-Hasting (MH) technique.

4.1 MCMC method

From (13), the conditional posterior distributions of α,β and θ are given respectively by

$$\pi_1^*(\alpha|\beta,\theta,y) \propto \alpha^{a_1+D_1+D_2-1}\;\theta^{\alpha D_2}e^{-b_1\alpha}$$
$$\times \prod_{j=1}^{2}\prod_{i=1}^{D_j} y_{ji}^{\alpha-1}\left[1+(\theta^{j-1}y_{ji})^{\alpha}\right]^{-(\beta+1)}\left[1+(\theta^{j-1}C_j)^{\alpha}\right]^{-\beta(n_j-D_j)}, \quad (15)$$

$$\pi_2^*(\beta|\alpha,\theta,y) \propto \beta^{a_2+D_1+D_2-1}\;e^{-b_2\beta}$$
$$\times \prod_{j=1}^{2}\prod_{i=1}^{D_j}\left[1+(\theta^{j-1}y_{ji})^{\alpha}\right]^{-(\beta+1)}\left[1+(\theta^{j-1}C_j)^{\alpha}\right]^{-\beta(n_j-D_j)}, \quad (16)$$

$$\pi_3^*(\theta|\alpha,\beta,y) \propto \theta^{\alpha D_2-1}\prod_{i=1}^{D_2}[1+(\theta y_{2i})^{\alpha}]^{-(\beta+1)}[1+(\theta C_2)^{\alpha}]^{-\beta(n_2-D_2)}. \quad (17)$$

Now, to obtain the Bayesian estimates and the corresponding credible intervals, we consider the following algorithm with normal proposal distribution, to generate random numbers for parameters α,β and θ from the posterior density function.

Algorithm

1. Start with initial points $\alpha^{(0)}, \beta^{(0)}$ and $\theta^{(0)}$ for α, β and θ respectively.
2. Set $j = 1$.
3. Generate $\alpha^{(j)}, \beta^{(j)}$ and $\theta^{(j)}$ from Eqs. (15), (16) and (17) with $N(\alpha^{(j-1)}, \sigma_1^2)$, $N(\beta^{(j-1)}, \sigma_2^2)$ and $N(\theta^{(j-1)}, \sigma_3^2)$ as proposal distribution, where σ_1^2, σ_2^2 and σ_3^2 are the variance of α, β and θ obtained using variance-covariance matrix.
4. Calculate $\psi^{(j)} = \psi(\alpha^{(j)}, \beta^{(j)}, \theta^{(j)})$.
5. Set $j = j + 1$.
6. Repeat Steps 3–5 N times.
7. Obtain the Bayesian estimates of the function ψ, under SEL as the following

$$\hat{\psi}_{BS} = IE(\psi|\mathbf{y}) = \frac{1}{N-M} \sum_{j=M+1}^{N} \psi^{(j)},$$

where M is the burn-in period.

8. The $100(1-\nu)\%$ credible intervals $(\Omega_{(N-M)\frac{\nu}{2}}, \Omega_{(N-M)_{1-\frac{\nu}{2}}})$ for Ω, can be obtained by ordering $\Omega^{(j)}$, $j = M+1, \ldots, N$ as $\Omega_1, \ldots, \Omega_{N-M}$, where Ω stands for α, β or θ. This interval is then given by

$$(\Omega_{[(N-M)\frac{\nu}{2}]}, \ldots, \Omega_{[(N-M)(1-\frac{\nu}{2})]}).$$

4 Simulation Study

In this section, in order to access the performance of the MLEs of the unknown parameters and the accelerated factor and compare it with its corresponding Bayesian estimates, we carry out a simulation study using different level of stress and different choices of k_j, r_j, T_{1j} values and fixed T_{2j} value. In computing the estimates, first we generate α and β from Gamma (a_1, b_1) and Gamma (a_2, b_2) prior density, respectively. We chose $a_1 = 2, b_1 = 1, a_2 = 1$ and $b_2 = 1$, these generated values are $\alpha_0 = 1.85078, \beta_0 = 0.780747$ in Table 3. Also we chose $a_1 = 1.5, b_1 = 1$, $a_2 = 2$ and $b_2 = 1$, these generated values are $\alpha_0 = 2.57433, \beta_0 = 0.99038$ in Table 4. Second, we generate 1000 samples from the Burr type XII distribution with $\alpha_0 = 1.85078, \beta_0 = 0.780747, \theta = 1.15$ and $\alpha = 2.57433, \beta = 0.99038, \theta = 1.55$. For MCMC technique , we set $N = 11000$ and $M = 1000$, when we apply the previous algorithm. The average estimate of $\psi^\star$ and the associated mean squared error (MSEs) are computed, respectively, as:

$$Average = \frac{1}{1000}\sum_{i=1}^{1000} \psi_i^\star, \qquad MSE = \frac{1}{1000}\sum_{i=1}^{1000} (\psi_i^\star - \psi)^2,$$

where $\psi_i^\star$ stands for an estimator (ML or Bayes) of α, β and θ, at the i^{th} iteration; and ψ stands for $\alpha_0 = 1.85078$, $\beta_0 = 0.780747$, $\theta_0 = 1.15$ and $\alpha_0 = 2.57433$, $\beta_0 = 0.99038$, $\theta_0 = 1.55$. The computational results are displayed in Tables 3–7, where average estimates of the parameters and their associated MSEs, are reported in Tables 3 and 4. Also 95% confidence intervals are computed and reported in Tables 5 and 7.

6 Concluding Remarks

(1) In this article, the MLE and Bayesian estimates of the parameters and the accelerated factor of the Burr type XII distribution using tampered random variable based on a given U-HCS under CS-PALT are obtained.
(2) Bayesian estimates have been obtained under SEL. Bayesian estimates cannot be obtained in a simple closed form. Therefor, MCMC method has been used to obtain it.
(3) We have also constructed approximate credible intervals for the parameters and accelerated factor.
(4) It has been noticed from Tables 3–7, that

i) The MSEs and confidence intervals length of all estimates (ML or Bayes) decrease as n_1 equal n_2.
ii) In most cases, Bayes estimates for α, β and θ better than MLEs.
iii) In most cases, credible intervals length for the parameters is smaller than approximate intervals length.
iv) In Tables 3–7, we could not confirm which better if T_{1j} was changed and T_{2j} was fixed. But in general increasing of T_{1j} is the best in some cases and for some parameters.

References

[1] Abdel-Hamid, A. H. (2009). Constant-partially accelerated life tests for Burr type-XII distribution with progressive Type-II censoring. Computational Statistics Data Analysis, 53(7), 2511-2523. doi:10.1016/j.csda.2009.01.018

[2] Ahmad, A. A., Soliman, A. A. and Yousef, M. M. (2016). Bayesian estimation of exponentiated Weibull distribution under partially acceleration Life tests. Bulletin of the Malaysian Mathematical Sciences Society, 39(1), 227-244. doi:10.1007/s40840-015-0170-9

[3] Balakrishnan, N., Rasouli, A. and Sanjari Farsipour, N. (2008). Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution. Journal of Statistical Computation and Simulation, 78(5), 475–488. doi:10.1080/00949650601158336

[4] Burr, I. W. (1942). Cumulative frequency functions. The Annals of Mathematical Statistics, 13, 215–232.

[5] Cheng, Y. F. and Wang, F. K. (2012). Estimating the Burr XII parameters in constant-stress partially accelerated life tests under multiple censored data. Communications in Statistics - Simulation and Computation, 41(9), 1711–1727. doi:10.1080/03610918.2011.617478

[6] DeGroot, M. H. and Goel, P. K. (1979). Bayesian estimation and optimal designs in partially accelerated life testing. Naval Research Logistics Quarterly, 26, 223–235.

[7] Habibi Rad, A. and Izanlo, M. (2012). An EM algorithm for estimating the parameters of the generalized exponential distribution under unified hybrid censored data. Journal of Statistical Research of Iran, 8(2), 215–228. doi:10.18869/acadpub.jsri.8.2.215

[8] Habibi Rad, A. and Yousefzadeh, F. (2014). Inference based on unified hybrid censored data for Weibull distribution. Paper presented at the 8th Annual International Conference on Statistics, 30 June 1–3 July 2014, Athens, Greece.

[9] Nelson, W. (1990). Accelerated testing: statistical models, test plans and data analysis. Wiley, New York, USA.

[10] Panahi, H. (2016). Estimation of the Burr type III distribution with application in unified hybrid censored sample of fracture toughness. Journal of Applied Statistics, 1–18. doi:10.1080/02664763.2016.1258549

[11] Panahi, H. and Sayyareh, A. (2016). Estimation and prediction for a unified hybrid-censored Burr type XII distribution. Journal of Statistical Computation and Simulation, 86(1), 55–73. doi:10.1080/00949655.2014.993985

[12] Soliman, A. A., Abd Ellah, A. H., Abou-Elheggag, N. A. and Modhesh, A. A. (2011). Bayesian inference and prediction of Burr type XII distribution for progressive first failure censored sampling, Intelligent Information Management, **3**(05), 175. doi:10.4236/iim.2011.35021

Table 3: Average estimates of the parameters and their associated MSEs at ($\alpha = 1.85078, \beta = 0.780747, \theta = 1.15$) and ($T_{12} = 4.63, T_{22} = 4.05$).

Case	$(k_1, k_2, r_1, r_2, n_1, n_2)$		(T_{11}, T_{21}) (1.08, 0.94) MLE (MSE)	BS (MSE)	(T_{11}, T_{21}) (2, 1.73) MLE (MSE)	BS (MSE)
I	(5, 15, 8, 24, 20, 60)	α	1.9189 (0.0818)	1.8989 (0.0713)	1.9191 (0.0825)	1.9102 (0.0719)
		β	0.8931 (0.0650)	0.8979 (0.0561)	0.7946 (0.0486)	0.8093 (0.0424)
		θ	1.1436 (0.0622)	1.1989 (0.0677)	1.2011 (0.0947)	1.2386 (0.0953)
	(10, 10, 16, 16, 40, 40)	α	1.9311 (0.0793)	1.9027 (0.0683)	1.9190 (0.0745)	1.9054 (0.0670)
		β	0.8609 (0.0319)	0.8594 (0.0294)	0.7852 (0.0263)	0.7908 (0.0242)
		θ	1.1789 (0.0504)	1.2238 (0.0613)	1.2029 (0.0813)	1.2413 (0.0900)
II	(5, 15, 16, 48, 20, 60)	α	1.9329 (0.0792)	1.9272 (0.0693)	1.9530 (0.0869)	1.9441 (0.0751)
		β	0.8162 (0.0438)	0.8303 (0.0389)	0.7413 (0.0266)	0.7608 (0.0222)
		θ	1.1623 (0.0731)	1.1951 (0.0707)	1.2328 (0.0786)	1.2617 (0.0791)
	(10, 10, 32, 32, 40, 40)	α	1.9177 (0.0607)	1.9085 (0.0548)	1.9115 (0.0667)	1.9015 (0.0601)
		β	0.7982 (0.0232)	0.8043 (0.0214)	0.7681 (0.0174)	0.7756 (0.0158)
		θ	1.1845 (0.0663)	1.2200 (0.0719)	1.1896 (0.0700)	1.2262 (0.0731)
III	(15, 15, 19, 57, 20, 60)	α	1.9299 (0.0755)	1.9210 (0.0650)	1.9277 (0.0909)	1.9182 (0.0771)
		β	0.7280 (0.0265)	0.7483 (0.0221)	0.7228 (0.0306)	0.7427 (0.0255)
		θ	1.2460 (0.0874)	1.2697 (0.0888)	1.2618 (0.0948)	1.2863 (0.0958)
	(10, 10, 38, 38, 40, 40)	α	1.8855 (0.0606)	1.8775 (0.0552)	1.8951 (0.0648)	1.8871 (0.0584)
		β	0.7496 (0.0181)	0.7587 (0.0162)	0.7497 (0.0186)	0.7585 (0.0167)
		θ	1.2008 (0.0681)	1.2333 (0.0739)	1.1957 (0.0728)	1.2280 (0.0781)
IV	(15, 45, 16, 48, 20, 60)	α	1.9201 (0.0813)	1.9142 (0.0709)	1.9544 (0.0833)	1.9440 (0.0704)
		β	0.8169 (0.0447)	0.8318 (0.0399)	0.7112 (0.0249)	0.7327 (0.0198)
		θ	1.1645 (0.0815)	1.1965 (0.0775)	1.2407 (0.0731)	1.2676 (0.0749)
	(30, 30, 32, 32, 40, 40)	α	1.9240 (0.0647)	1.9139 (0.0580)	1.9160 (0.0628)	1.9054 (0.0564)
		β	0.7986 (0.0255)	0.8051 (0.0236)	0.7360 (0.0162)	0.7447 (0.0143)
		θ	1.1844 (0.0706)	1.2191 (0.0765)	1.1832 (0.0565)	1.2172 (0.0615)
V	(15, 45, 19, 57, 20, 60)	α	1.9322 (0.0841)	1.9232 (0.0724)	1.9475 (0.0851)	1.9347 (0.0719)
		β	0.7250 (0.0283)	0.7451 (0.0237)	0.6739 (0.0306)	0.6965 (0.0241)
		θ	1.2510 (0.0876)	1.2752 (0.0893)	1.2807 (0.0858)	1.3021 (0.0877)
	(30, 30, 38, 38, 40, 40)	α	1.8886 (0.0587)	1.8811 (0.0533)	1.8836 (0.0600)	1.8755 (0.0542)
		β	0.7483 (0.0193)	0.7570 (0.0174)	0.7208 (0.0168)	0.7304 (0.0146)
		θ	1.2014 (0.0716)	1.2342 (0.0773)	1.1799 (0.0494)	1.2125 (0.0532)
VI	(19, 57, 20, 60, 20, 60)	α	1.9412 (0.1074)	1.9348 (0.0786)	1.9412 (0.1074)	1.9348 (0.0786)
		β	0.7272 (0.0430)	0.7436 (0.0236)	0.7272 (0.0430)	0.7436 (0.0236)
		θ	1.2390 (0.0992)	1.2630 (0.0834)	1.2390 (0.0992)	1.2630 (0.0834)
	(38, 38, 40, 40, 40, 40)	α	1.8827 (0.0665)	1.8752 (0.0594)	1.8827 (0.0665)	1.8752 (0.0594)
		β	0.7612 (0.0191)	0.7702 (0.0173)	0.7612 (0.0191)	0.7702 (0.0173)
		θ	1.1939 (0.0704)	1.2244 (0.0747)	1.1939 (0.0704)	1.2244 (0.0747)

Table 4: Average estimates of the parameters and their associated MSEs at ($\alpha = 2.57433, \beta = 0.99038, \theta = 1.55$) and ($T_{12} = 2.27, T_{22} = 1.48$).

Case	$(k_1, k_2, r_1, r_2, n_1, n_2)$		(T_{11}, T_{21}) (0.92, 0.60) MLE (MSE)	(T_{11}, T_{21}) (0.92, 0.60) BS (MSE)	(T_{11}, T_{21}) (1.38, 0.89) MLE (MSE)	(T_{11}, T_{21}) (1.38, 0.89) BS (MSE)
I	(5, 15,	α	2.6565 (0.1279)	2.5563 (0.1021)	2.6737 (0.1290)	2.5782 (0.0951)
	8, 24,	β	1.1693 (0.1116)	1.2427 (0.1262)	1.0098 (0.0727)	1.0884 (0.0732)
	20, 60)	θ	1.5140 (0.0443)	1.4875 (0.0404)	1.5827 (0.0775)	1.5459 (0.0620)
	(10, 10,	α	2.6643 (0.1618)	2.5678 (0.1301)	2.6478 (0.1059)	2.5667 (0.0859)
	16, 16,	β	1.1117 (0.0547)	1.1427 (0.0586)	1.0054 (0.0411)	1.0439 (0.0407)
	40, 40)	θ	1.5599 (0.0390)	1.5502 (0.0393)	1.5757 (0.0585)	1.5647 (0.0552)
II	(5, 15,	α	2.6839 (0.1357)	2.5963 (0.0968)	2.7096 (0.1457)	2.6114 (0.0979)
	16, 48,	β	1.0629 (0.0793)	1.1380 (0.0867)	0.9522 (0.0402)	1.0322 (0.0358)
	20, 60)	θ	1.5423 (0.0636)	1.5106 (0.0537)	1.6080 (0.0580)	1.5682 (0.0449)
	(10, 10,	α	2.6623 (0.0987)	2.5891 (0.0776)	2.6389 (0.1049)	2.5642 (0.0842)
	32, 32,	β	1.0314 (0.0380)	1.0694 (0.0399)	0.9897 (0.0272)	1.0298 (0.0267)
	40, 40)	θ	1.5479 (0.0476)	1.5387 (0.0451)	1.5646 (0.0494)	1.5531 (0.0459)
III	(15, 15,	α	2.6692 (0.1269)	2.5713 (0.0887)	2.6747 (0.1400)	2.5752 (0.0975)
	19, 57,	β	0.9255 (0.0412)	1.0042 (0.0338)	0.9210 (0.0457)	0.9994 (0.0370)
	20, 60)	θ	1.6199 (0.0636)	1.5771 (0.0493)	1.6271 (0.0712)	1.5834 (0.0551)
	(10, 10,	α	2.6102 (0.0930)	2.5373 (0.0774)	2.6227 (0.1027)	2.5494 (0.0842)
	38, 38,	β	0.9640 (0.0261)	1.0056 (0.0238)	0.9584 (0.0259)	0.9999 (0.0232)
	40, 40)	θ	1.5699 (0.0517)	1.5565 (0.0481)	1.5691 (0.0540)	1.5558 (0.0500)
IV	(15, 45,	α	2.6746 (0.1325)	2.5876 (0.0950)	2.6997 (0.1406)	2.5994 (0.0968)
	16, 48,	β	1.0593 (0.0732)	1.1343 (0.0808)	0.9102 (0.0373)	0.9908 (0.0274)
	20, 60)	θ	1.5401 (0.0621)	1.5091 (0.0531)	1.6243 (0.0540)	1.5817 (0.0406)
	(30, 30,	α	2.6692 (0.1010)	2.5947 (0.0781)	2.6300 (0.0912)	2.5547 (0.0752)
	32, 32,	β	1.0171 (0.0350)	1.0562 (0.0361)	0.9503 (0.0196)	0.9915 (0.0166)
	40, 40)	θ	1.5646 (0.0530)	1.5542 (0.0499)	1.5575 (0.0380)	1.5454 (0.0352)
V	(15, 45,	α	2.6759 (0.1353)	2.5774 (0.0922)	2.6637 (0.1402)	2.5605 (0.0977)
	19, 57,	β	0.9242 (0.0374)	1.0032 (0.0300)	0.8708 (0.0403)	0.9507 (0.0245)
	20, 60)	θ	1.6289 (0.0656)	1.5856 (0.0499)	1.6517 (0.0595)	1.6039 (0.0421)
	(30, 30,	α	2.6053 (0.1010)	2.5329 (0.0843)	2.6166 (0.1002)	2.5410 (0.0825)
	38, 38,	β	0.9647 (0.0264)	1.0062 (0.0243)	0.9165 (0.0241)	0.9589 (0.0183)
	40, 40)	θ	1.5642 (0.0524)	1.5510 (0.0488)	1.5682 (0.0417)	1.5538 (0.0385)
VI	(19, 57,	α	2.6580 (0.1303)	2.5619 (0.0931)	2.6580 (0.1303)	2.5619 (0.0931)
	20, 60,	β	0.9342 (0.0414)	1.0102 (0.0356)	0.9342 (0.0414)	1.0102 (0.0356)
	20, 60)	θ	1.6197 (0.0653)	1.5775 (0.0522)	1.6197 (0.0653)	1.5775 (0.0522)
	(38, 38,	α	2.6409 (0.1011)	2.5672 (0.0799)	2.6409 (0.1011)	2.5672 (0.0799)
	40, 40,	β	0.9547 (0.0257)	0.9968 (0.0228)	0.9547 (0.0257)	0.9968 (0.0228)
	40, 40)	θ	1.5801 (0.0544)	1.5647 (0.0499)	1.5801 (0.0544)	1.5647 (0.0499)

Table 5: 95% confidence intervals for the parameters $\alpha = 1.85078$, $\beta = 0.780747$ and $\theta = 1.15$.

Case	(n_1, n_2)		$(T_{11}, T_{21}) = (1.08, 0.94)$ Approximate Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (1.08, 0.94)$ Credible Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (2, 1.73)$ Approximate Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (2, 1.73)$ Credible Intervals L, U (*Length*)
I	(20, 60)	α	1.3647, 2.4731 (1.1084)	1.3962, 2.4835 (1.0873)	1.3925, 2.4456 (1.0531)	1.4469, 2.4751 (1.0282)
		β	0.3535, 1.4326 (1.0791)	0.4704, 1.4853 (1.0149)	0.3767, 1.2125 (0.8359)	0.4656, 1.2717 (0.8061)
		θ	0.5388, 1.7484 (1.2096)	0.6902, 1.9937 (1.3034)	0.6085, 1.7937 (1.1852)	0.7399, 1.9585 (1.2186)
	(40, 40)	α	1.3977, 2.4644 (1.0667)	1.4147, 2.4581 (1.0434)	1.4386, 2.3993 (0.9607)	1.4678, 2.4129 (0.9451)
		β	0.4844, 1.2374 (0.7531)	0.5348, 1.2651 (0.7304)	0.4829, 1.0876 (0.6046)	0.5248, 1.1178 (0.5930)
		θ	0.6420, 1.7158 (1.0738)	0.7433, 1.9387 (1.1954)	0.6740, 1.7318 (1.0578)	0.7761, 1.9046 (1.1285)
II	(20, 60)	α	1.4177, 2.4480 (1.0304)	1.4746, 2.4815 (1.0068)	1.4144, 2.4916 (1.0772)	1.4765, 2.5233 (1.0469)
		β	0.4085, 1.2238 (0.8153)	0.4930, 1.2815 (0.7885)	0.3682, 1.1143 (0.7461)	0.4503, 1.1791 (0.7288)
		θ	0.6187, 1.7060 (1.0873)	0.7312, 1.8411 (1.1100)	0.6523, 1.8134 (1.1612)	0.7669, 1.9472 (1.1803)
	(40, 40)	α	1.4524, 2.3831 (0.9308)	1.4853, 2.4029 (0.9176)	1.4420, 2.3810 (0.9390)	1.4757, 2.3993 (0.9236)
		β	0.5018, 1.0945 (0.5927)	0.5422, 1.1251 (0.5829)	0.4816, 1.0547 (0.5731)	0.5225, 1.0869 (0.5644)
		θ	0.6739, 1.6951 (1.0213)	0.7705, 1.8497 (1.0793)	0.6718, 1.7074 (1.0355)	0.7702, 1.8633 (1.0931)
III	(20, 60)	α	1.3983, 2.4615 (1.0631)	1.4606, 2.4915 (1.0310)	1.3913, 2.4640 (1.0728)	1.4554, 2.4944 (1.0390)
		β	0.3808, 1.0752 (0.6944)	0.4558, 1.1397 (0.6839)	0.3771, 1.0685 (0.6914)	0.4523, 1.1333 (0.6811)
		θ	0.6776, 1.8143 (1.1366)	0.7794, 1.9317 (1.1523)	0.6822, 1.8415 (1.1593)	0.7863, 1.9628 (1.1765)
	(40, 40)	α	1.4198, 2.3513 (0.9316)	1.4586, 2.3731 (0.9145)	1.4274, 2.3629 (0.9355)	1.4665, 2.3840 (0.9175)
		β	0.4755, 1.0237 (0.5482)	0.5164, 1.0568 (0.5404)	0.4761, 1.0234 (0.5474)	0.5163, 1.0571 (0.5408)
		θ	0.6731, 1.7286 (1.0555)	0.7695, 1.8762 (1.1067)	0.6742, 1.7171 (1.0429)	0.7696, 1.8649 (1.0954)
IV	(20, 60)	α	1.4061, 2.4340 (1.0280)	1.4631, 2.4684 (1.0053)	1.4066, 2.5022 (1.0955)	1.4704, 2.5334 (1.0630)
		β	0.4073, 1.2265 (0.8192)	0.4921, 1.2851 (0.7929)	0.3513, 1.0711 (0.7197)	0.4328, 1.1379 (0.7051)
		θ	0.6123, 1.7168 (1.1046)	0.7267, 1.8531 (1.1265)	0.6499, 1.8316 (1.1816)	0.7658, 1.9624 (1.1966)
	(40, 40)	α	1.4566, 2.3915 (0.9348)	1.4892, 2.4097 (0.9205)	1.4395, 2.3925 (0.9530)	1.4749, 2.4121 (0.9372)
		β	0.5023, 1.0948 (0.5925)	0.5433, 1.1260 (0.5827)	0.4589, 1.0131 (0.5543)	0.5001, 1.0462 (0.5460)
		θ	0.6758, 1.6931 (1.0173)	0.7710, 1.8466 (1.0756)	0.6625, 1.7038 (1.0413)	0.7604, 1.8589 (1.0985)
V	(20, 60)	α	1.3965, 2.4678 (1.0713)	1.4600, 2.5007 (1.0407)	1.3930, 2.5020 (1.1091)	1.4592, 2.5315 (1.0722)
		β	0.3788, 1.0711 (0.6923)	0.4536, 1.1362 (0.6826)	0.3477, 1.0000 (0.6523)	0.4213, 1.0669 (0.6455)
		θ	0.6792, 1.8228 (1.1436)	0.7816, 1.9415 (1.1599)	0.6882, 1.8732 (1.1850)	0.7925, 1.9946 (1.2021)

Table 5: Continued

Case	(n_1, n_2)		$(T_{11},T_{21})=(1.08,0.94)$ Approximate Intervals L, U ($Length$)	$(T_{11},T_{21})=(1.08,0.94)$ Credible Intervals L, U ($Length$)	$(T_{11},T_{21})=(2,1.73)$ Approximate Intervals L, U ($Length$)	$(T_{11},T_{21})=(2,1.73)$ Credible Intervals L, U ($Length$)
	(40, 40)	α	1.4216, 2.3555 (0.9339)	1.4615, 2.3780 (0.9165)	1.4135, 2.3538 (0.9403)	1.4535, 2.3761 (0.9226)
		β	0.4752, 1.0215 (0.5463)	0.5160, 1.0544 (0.5384)	0.4549, 0.9867 (0.5318)	0.4954, 1.0204 (0.5250)
		θ	0.6764, 1.7263 (1.0500)	0.7724, 1.8723 (1.0999)	0.6556, 1.7042 (1.0485)	0.7532, 1.8521 (1.0989)
VI	(20, 60)	α	1.4010, 2.4815 (1.0805)	1.4691, 2.5158 (1.0467)	1.4010, 2.4815 (1.0805)	1.4691, 2.5158 (1.0467)
		β	0.3925, 1.0620 (0.6695)	0.4609, 1.1196 (0.6587)	0.3925, 1.0620 (0.6695)	0.4609, 1.1196 (0.6587)
		θ	0.6884, 1.7896 (1.1012)	0.7833, 1.8997 (1.1164)	0.6884, 1.7896 (1.1012)	0.7833, 1.8997 (1.1164)
	(40, 40)	α	1.4214, 2.3441 (0.9226)	1.4608, 2.3659 (0.9051)	1.4214, 2.3441 (0.9226)	1.4608, 2.3659 (0.9051)
		β	0.4913, 1.0312 (0.5400)	0.5304, 1.0647 (0.5344)	0.4913, 1.0312 (0.5400)	0.5304, 1.0647 (0.5344)
		θ	0.6793, 1.7085 (1.0292)	0.7701, 1.8473 (1.0772)	0.6793, 1.7085 (1.0292)	0.7701, 1.8473 (1.0772)

Table 6: 95% confidence intervals for the parameters $\alpha=2.57433$, $\beta=0.99038$ and $\theta=1.55$.

Case	(n_1, n_2)		$(T_{11},T_{21})=(0.92,0.60)$ Approximate Intervals L, U ($Length$)	$(T_{11},T_{21})=(0.92,0.60)$ Credible Intervals L, U ($Length$)	$(T_{11},T_{21})=(1.38,0.89)$ Approximate Intervals L, U ($Length$)	$(T_{11},T_{21})=(1.38,0.89)$ Credible Intervals L, U ($Length$)
I	(20, 60)	α	1.9206, 3.3924 (1.4718)	1.8954, 3.3073 (1.4120)	2.0003, 3.3472 (1.3469)	1.9913, 3.2615 (1.2702)
		β	0.4636, 1.8751 (1.4115)	0.6687, 2.0234 (1.3547)	0.4923, 1.5274 (1.0351)	0.6414, 1.6712 (1.0298)
		θ	0.9741, 2.0539 (1.0798)	1.0307, 2.1085 (1.0777)	1.0532, 2.1121 (1.0589)	1.0992, 2.1295 (1.0303)
	(40, 40)	α	1.9437, 3.3849 (1.4412)	1.9170, 3.3018 (1.3847)	2.0268, 3.2689 (1.2421)	2.0048, 3.2009 (1.1962)
		β	0.6175, 1.6060 (0.9885)	0.7150, 1.6804 (0.9655)	0.6331, 1.3777 (0.7447)	0.7073, 1.4504 (0.7431)
		θ	1.0729, 2.0468 (0.9738)	1.1026, 2.1354 (1.0328)	1.1096, 2.0418 (0.9321)	1.1432, 2.1027 (0.9595)
II	(20, 60)	α	2.0350, 3.3328 (1.2978)	2.0333, 3.2567 (1.2234)	2.0329, 3.3862 (1.3533)	2.0332, 3.2977 (1.2645)
		β	0.5465, 1.5793 (1.0328)	0.6919, 1.7190 (1.0271)	0.4870, 1.4174 (0.9304)	0.6257, 1.5625 (0.9368)
		θ	1.0521, 2.0324 (0.9803)	1.0912, 2.0438 (0.9526)	1.0908, 2.1253 (1.0345)	1.1266, 2.1279 (1.0013)

Table 6: Continued

Case	(n_1, n_2)		$(T_{11}, T_{21}) = (0.92, 0.60)$ Approximate Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (0.92, 0.60)$ Credible Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (1.38, 0.89)$ Approximate Intervals L, U (*Length*)	$(T_{11}, T_{21}) = (1.38, 0.89)$ Credible Intervals L, U (*Length*)
	(40, 40)	α	2.0671, 3.2575 (1.1903)	2.0514, 3.1975 (1.1461)	2.0419, 3.2358 (1.1939)	2.0274, 3.1752 (1.1478)
		β	0.6673, 1.3956 (0.7283)	0.7393, 1.4664 (0.7272)	0.6375, 1.3418 (0.7043)	0.7102, 1.4150 (0.7048)
		θ	1.1067, 1.9892 (0.8825)	1.1392, 2.0403 (0.9011)	1.1078, 2.0213 (0.9135)	1.1414, 2.0718 (0.9305)
III	(20, 60)	α	2.0023, 3.3360 (1.3338)	2.0065, 3.2488 (1.2423)	2.0035, 3.3460 (1.3425)	2.0081, 3.2549 (1.2468)
		β	0.4955, 1.3554 (0.8599)	0.6243, 1.5006 (0.8763)	0.4924, 1.3495 (0.8572)	0.6212, 1.4934 (0.8722)
		θ	1.1111, 2.1286 (1.0175)	1.1343, 2.1255 (0.9912)	1.1140, 2.1403 (1.0263)	1.1385, 2.1347 (0.9962)
	(40, 40)	α	2.0236, 3.1967 (1.1731)	2.0139, 3.1377 (1.1238)	2.0325, 3.2129 (1.1804)	2.0227, 3.1545 (1.1318)
		β	0.6301, 1.2979 (0.6678)	0.7013, 1.3728 (0.6715)	0.6262, 1.2906 (0.6644)	0.6968, 1.3653 (0.6685)
		θ	1.1100, 2.0298 (0.9198)	1.1412, 2.0749 (0.9338)	1.1112, 2.0269 (0.9157)	1.1426, 2.0720 (0.9294)
IV	(20, 60)	α	2.0276, 3.3216 (1.2940)	2.0261, 3.2468 (1.2207)	2.0180, 3.3815 (1.3635)	2.0190, 3.2898 (1.2708)
		β	0.5450, 1.5736 (1.0287)	0.6901, 1.7125 (1.0224)	0.4652, 1.3552 (0.8900)	0.6022, 1.5008 (0.8987)
		θ	1.0480, 2.0322 (0.9843)	1.0883, 2.0441 (0.9559)	1.0967, 2.1520 (1.0553)	1.1314, 2.1491 (1.0177)
	(40, 40)	α	2.0697, 3.2686 (1.1989)	2.0542, 3.2068 (1.1527)	2.0305, 3.2294 (1.1989)	2.0164, 3.1685 (1.1522)
		β	0.6567, 1.3775 (0.7208)	0.7292, 1.4490 (0.7199)	0.6100, 1.2907 (0.6807)	0.6822, 1.3638 (0.6816)
		θ	1.1166, 2.0126 (0.8960)	1.1495, 2.0627 (0.9132)	1.0961, 2.0189 (0.9228)	1.1310, 2.0699 (0.9389)
V	(20, 60)	α	2.0067, 3.3451 (1.3384)	2.0108, 3.2561 (1.2453)	1.9832, 3.3442 (1.3610)	1.9898, 3.2492 (1.2593)
		β	0.4939, 1.3546 (0.8607)	0.6228, 1.4999 (0.8770)	0.4611, 1.2806 (0.8195)	0.5881, 1.4261 (0.8380)
		θ	1.1164, 2.1415 (1.0251)	1.1399, 2.1361 (0.9962)	1.1192, 2.1841 (1.0648)	1.1442, 2.1760 (1.0318)
	(40, 40)	α	2.0201, 3.1906 (1.1705)	2.0103, 3.1331 (1.1228)	2.0198, 3.2134 (1.1936)	2.0103, 3.1522 (1.1419)
		β	0.6305, 1.2988 (0.6683)	0.7012, 1.3737 (0.6725)	0.5951, 1.2379 (0.6428)	0.6662, 1.3133 (0.6471)
		θ	1.1053, 2.0230 (0.9177)	1.1370, 2.0690 (0.9320)	1.1013, 2.0350 (0.9337)	1.1338, 2.0805 (0.9467)
VI	(20, 60)	α	1.9957, 3.3203 (1.3246)	2.0023, 3.2352 (1.2329)	1.9957, 3.3203 (1.3246)	2.0023, 3.2352 (1.2329)
		β	0.5179, 1.3505 (0.8326)	0.6398, 1.4901 (0.8502)	0.5179, 1.3505 (0.8326)	0.6398, 1.4901 (0.8502)
		θ	1.1212, 2.1182 (0.9970)	1.1388, 2.1124 (0.9736)	1.1212, 2.1182 (0.9970)	1.1388, 2.1124 (0.9736)
	(40, 40)	α	2.0501, 3.2318 (1.1817)	2.0429, 3.1720 (1.1290)	2.0501, 3.2318 (1.1817)	2.0429, 3.1720 (1.1290)
		β	0.6322, 1.2773 (0.6451)	0.7021, 1.3516 (0.6495)	0.6322, 1.2773 (0.6451)	0.7021, 1.3516 (0.6495)
		θ	1.1265, 2.0337 (0.9072)	1.1533, 2.0715 (0.9182)	1.1265, 2.0337 (0.9072)	1.1533, 2.0715 (0.9182)