Delta and Theta Generalized Closed Sets in Bitopological Spaces

F. H. Khedr

Department of Mathematics-Faculty of Sciences-University of Assiut- Assiut 715161- Egypt
E-mail: Khedrfathi@gmail.com

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In this paper we introduce new two types of generalized closed sets in bitopological spaces and study some of it's properties and it's relations with other kinds of generalized closed sets. Using these sets we obtain a characterization of some separation axioms in bitopological spaces.

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1- Introduction and preliminaries

Throughout this paper, $(X,\tau_1,\tau_2)$ (or simply $X$) always mean a bitopological space (or simply a space) on which no separation axioms are assumed unless explicitly stated. Also, $i,j \in \{1,2\}$ and $i \neq j$. Let $A$ be a subset of a space $(X,\tau_1,\tau_2)$. The closure of $A$ and the interior of $A$ in the topological space $(X,\tau_i)$ are denoted by $i - \text{Cl}(A)$ and $i - \text{Int}(A)$, respectively. We write $i - \text{open}$ (resp. $i - \text{closed}$) set to mean that the set is open (resp. closed) in the topological space $(X,\tau_i)$.

Let $(X,\tau_1,\tau_2)$ be a bitopological space. A point $x \in X$ is said to be in the $ij - \delta$ - closure [1] (resp. $ij - \theta$ - closure [1]) of a subset $A$ of $X$ if $i - \text{Int}(j - \text{Cl}(U)) \cap A \neq \phi$ (resp. $j - \text{Cl}(U) \cap A \neq \phi$) for every $i - \text{open}$ set $U$ containing $x$. The $ij - \delta$ - closure (resp. $ij - \theta$ - closure) of a subset $A$ is denoted by $\text{Cl}_ij^\delta(A)$ (resp. $\text{Cl}_ij^\theta(A)$). A subset $A \subset X$ is called $ij - \delta$ - closed (resp. $ij - \theta$ - closed) if $A = \text{Cl}_ij^\delta(A)$ (resp. $A = \text{Cl}_ij^\theta(A)$). The complement of an $ij - \delta$ - closed (resp. $ij - \theta$ - closed) set is called $ij - \delta$ - open (resp. $ij - \theta$ - open). The set of all $ij - \delta$ - open (resp. $ij - \theta$ - open)
sets form a topology on \(X\) will be denote by \(\tau^\delta_i\) [7] (resp. \(\tau^\theta_i\) [7]). From the
definition it follows that \(\tau^\theta_i \subset \tau^\delta_i \subset \tau_i\) [7]. The space \((X, \tau^\delta_1, \tau^\delta_2)\) is called
the pairwise semi-regularization of \((X, \tau_1, \tau_2)\) [12]. A space \((X, \tau_1, \tau_2)\) is
pairwise semi-regular if \(\tau_i = \tau^\delta_i\) [12]. A bitopological space \((X, \tau_1, \tau_2)\) is
called pairwise regular [4] if for each \(x \in X\) and each \(i\) —closed set \(F\) not containing \(x\), there exist an \(i\) —open set \(U\) and a \(j\) —open sets \(V\) such that
\(x \in U\), \(F \subset V\) and \(U \cap V = \phi\), equivalently, if for each \(i\) —open set \(U\) and
\(x \in U\), there exists an \(i\) —open \(V\) such that \(x \in V \subset j\) —\(\text{Cl}(V) \subset U\) . A
space \((X, \tau_1, \tau_2)\) is pairwise regular if and only if \(\tau_i = \tau^\theta_i\) [16]. A subset \(A\) of a
bitopological space \((X, \tau_1, \tau_2)\) is called \(ij\) —regular open [17] (resp. \(ij\) —regular closed [17]) if
\((\text{Int}_i j \text{Cl}_i A) \subset (\text{Cl}_i j \text{Int}_i A)\). A is called \(ij\) —nowhere dense if
\((\text{Int}_i j \text{Cl}_i A) = \phi\). The family of all \(ij\) —regular open subsets of a
bitopological space \((X, \tau_1, \tau_2)\) form a base for a topology \(\tau^*_i\) on \(X\). It is
shown in [12] that \(\tau^*_i = \tau^\delta_i\).

**Remark 1.1.** \(\text{Cl}^\delta_{ij}(A)\) is the \(i\) —closure of \(A\) with respect to \((X, \tau_1^\delta, \tau_2^\delta)\). In
general, \(\text{Cl}^\theta_{ij}(A)\) will not be the \(i\) —closure of \(A\) with respect to\((X, \tau_1^\theta, \tau_2^\theta)\).

**Remark 1.2.[7]** \(A \subset i\) —\(\text{Cl}(A) \subset \text{Cl}^\delta_{ij}(A) \subset \text{Cl}^\theta_{ij}(A)\).

**Definition 1.3.** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called:

1. \(ij\) —\(\alpha\) —closed [5] if \(i\) —\(\text{Cl}(j\) —\(\text{Int}(i\) —\(\text{Cl}(A)) \subset A\).
2. \(ij\) —\(\alpha\) —open [5] if \(X \setminus A\) is \(ij\) —\(\alpha\) —closed or equivalently if
\(A \subset i\) —\(\text{Int}(j\) —\(\text{Cl}(i\) —\(\text{Int}(A)))\).
3. \(ij\) —semi closed [2] if \(j\) —\(\text{Int}(i\) —\(\text{Cl}(A)) \subset A\).
4. \(ij\) —semi open [2] if \(X \setminus A\) is \(ij\) —semi closed or equivalently if
\(A \subset j\) —\(\text{Cl}(i\) —\(\text{Int}(A))\).
5. \(ij\) —preclosed [5] if \(i\) —\(\text{Cl}(j\) —\(\text{Int}(A)) \subset A\).
6. \(ij\) —preopen [5] if \(X \setminus A\) is \(ij\) —preclosed or equivalently if
\(A \subset i\) —\(\text{Int}(j\) —\(\text{Cl}(A))\).
7. \(ij\) —\(\beta\) —closed [5] if \(j\) —\(\text{Int}(i\) —\(\text{Cl}(j\) —\(\text{Int}(A))) \subset A\).
(8) \( ij - \beta \)-open [5] if \( X \setminus A \) is \( ij - \beta \)-closed or equivalently if \( A \subset j - \text{Cl}(i - \text{Int}(j - \text{Cl}(A))) \).

(9) \( ij - g \)-closed [3] if \( j - \text{Cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( i - \) open.

(10) \( ij - \beta g \)-closed [9] if \( \text{Cl}_{ij}^\beta(A) \subset U \), whenever \( A \subset U \) and \( U \) is \( i - \) open.

For a subset \( A \) of a space \((X, \tau_1, \tau_2)\), the \( ij - \alpha \)-closure (resp. \( ij - \alpha \)-semiclosure, \( ij - \alpha \)-preclosure, \( ij - \beta \)-closure) of a set \( A \subset X \) is the smallest \( ij - \alpha \)-closed (resp. \( ij - \alpha \)-semiclosed, \( ij - \alpha \)-preclosed, \( ij - \beta \)-closed) set containing \( A \). These closures are denoted by \( ij - \alpha \text{Cl}(A) \) (resp. \( ij - s \text{Cl}(A) \), \( ij - p \text{Cl}(A) \), \( ij - \beta \text{Cl}(A) \)).

Lemma 1.4. [18] Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \subset X \), then

(i) \( ij - \alpha \text{Cl}(A) = A \cup i - \text{Cl}(j - \text{Int}(i - \text{Cl}(A))) \).

(ii) \( ij - p \text{Cl}(A) \supset A \cup i - \text{Cl}(j - \text{Int}(A)) \).

(iii) \( ij - s \text{Cl}(A) = A \cup j - \text{Int}(i - \text{Cl}(A)) \).

(iv) \( ij - \beta \text{Cl}(A) \supset A \cup j - \text{Int}(i - \text{Cl}(j - \text{Int}(A))) \).

2- \( ij - \delta \theta \)-closed and \( ij - \theta \delta \)-closed sets

Let \( P = \{\tau, \alpha, s, p, \beta, \delta, \theta\} \), where \( \tau \) denote the set of all \( i - \) open sets, \( \alpha \) denote the set of all \( ij - \alpha \)-open sets, \( s \) denote the set of all \( ij - \text{semi} \) open sets, \( p \) the set of all \( ij - \text{pre} \) open sets, \( \beta \) the set of all \( ij - \beta \)-open sets, \( \delta \) the set of all \( ij - \delta \)-open sets and \( \theta \) the set of all \( ij - \theta \)-open sets.

Definition 2.1. A subset \( A \subset X \) is called \( ij - qr \)-closed if \( ji - q \text{Cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( i - r \)-open, where \( r, q \in P \).

Remark 2.2. If \( r, q \in P \), then every \( ij - qr \)-closed subset of \((X, \tau_1, \tau_2)\) is \( ij - q \)-closed if and only if each singleton of \( X \) is either \( ij - q \)-open or \( ij - r \)-closed.

Definition 2.3. A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called:

(1) \( ij - \delta g \)-closed if \( \text{Cl}_{ij}^\delta(A) \subset U \), whenever \( A \subset U \) and \( U \) is \( i - \) open.

(2) \( ij - g \delta \)-closed if \( j - \text{Cl}(A) \subset U \), whenever \( A \subset U \) and \( U \) is \( ij - \delta \)-open.
(3) $ij - \delta g^* - \text{closed if } Cl_{ji}^\delta (A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $ij - \delta -$ open.

(4) $ij - \theta g - \text{closed if } j - Cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $ij - \theta -$ open.

(5) $ij - \theta g^* - \text{closed if } Cl_{ji}^\theta (A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $ij - \theta -$ open.

In the notation of Definition 2.1, we have:

(1) $ij - \delta g - \text{closed}$ is equivalent to $ij - \delta \tau - \text{closed}$.

(2) $ij - \theta g - \text{closed}$ is equivalent to $ij - \theta \tau - \text{closed}$.

(3) $ij - \delta g^* - \text{closed}$ is equivalent to $ij - \delta \delta - \text{closed}$.

(4) $ij - \theta g - \text{closed}$ is equivalent to $ij - \theta \tau - \text{closed}$.

(5) $ij - \theta g^* - \text{closed}$ is equivalent to $ij - \theta \theta - \text{closed}$.

(6) $ij - \theta g - \text{closed}$ is equivalent to $ij - \theta \theta - \text{closed}$.

(7) $ij - \theta g^* - \text{closed}$ is equivalent to $ij - \theta \theta - \text{closed}$.

**Definition 2.4.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called:

(1) $ij - \delta \theta - \text{closed}$, if $ji - \delta - Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $ij - \delta -$ open.

(2) $ij - \theta \delta - \text{closed}$, if $ji - \theta - Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $ij - \delta -$ open.

**Remark 2.5.** Obviously every $ji - \delta -$ closed set is $ij - \delta \theta -$ closed and every $ji - \theta -$ closed set is $ij - \theta \delta -$ closed. Since $\tau_i^\theta \subseteq \tau_i^\delta \subseteq \tau_i$, every $ij - \theta \delta -$ closed set is $ij - \delta \theta -$ closed. If $x \in U$ and $U$ is $ij - \theta -$ open, then there exists an $i -$ open set $V$ such that $x \in V$ and $j - Cl(V) \subseteq U$. Since $j - C(V)$ is $ji - \delta -$ closed, we have $ji - \delta - Cl(\{x\}) \subseteq U$, that is, every singleton in any space is always $ij - \delta \theta -$ closed. Let $X$ be an infinite set, $p \in X$ be a particular point and $\tau_1 = \tau_2 = \text{ the topology on } X \text{ consisting of } X \text{ and all subsets of } X \text{ not containing } p$. If $x \neq p$, then $\{x\}$ is not $ij - \delta -$ open and $j - Cl(\{x\}) = \{x, p\} \subseteq ji - \theta - Cl(\{x\})$. Thus $\{x\}$ is $ij - \delta \theta -$ closed but fails to be $ij - \theta \delta -$ closed.

**Definition 2.6.** A bitopological space $(X, \tau_1, \tau_2)$ is said to be satisfies property
(1) A, if every $ij - \delta\theta$-closed set is $ji - \delta$-closed, i.e., each singleton is either $ji - \delta$-open or $ij - \theta$-closed.

(2) B, if every $ij - \theta\delta$-closed set is $ji - \theta$-closed, i.e., each singleton is either $ji - \theta$-open or $ij - \delta$-closed.

Recall that $(X, \tau_1, \tau_2)$ is pairwise $T_\frac{1}{2}$ [3], if each singleton is either $j - \delta$-open or $i - \theta$-closed. $(X, \tau_1, \tau_2)$ is called pairwise weakly Hausdorff (resp. pairwise almost weakly Hausdorff) if $(X, \tau_1^\theta, \tau_2^\delta)$ is pairwise $T_1$ (resp. $T_{\frac{1}{2}}$).

One can observe that, $(X, \tau_1^\theta, \tau_2^\delta)$ is pairwise Hausdorff if and only if $(X, \tau_1^\theta, \tau_2^\theta)$ is pairwise $T_1$ if and only if $(X, \tau_1^\theta, \tau_2^\delta)$ is pairwise $T_{\frac{1}{2}}$ if and only if $(X, \tau_1^\theta, \tau_2^\theta)$ is pairwise $T_0$.

**Theorem 2.7.** For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent:

(a) $(X, \tau_1, \tau_2)$ is pairwise Hausdorff.

(b) $(X, \tau_1, \tau_2)$ satisfies A.

(c) $(X, \tau_1, \tau_2)$ is pairwise almost weakly Hausdorff and $ij - \delta$-closed singletons are $ij - \theta$-closed.

**Proof:** (a) $\Rightarrow$ (b) If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff, then $(X, \tau_1^\theta, \tau_2^\theta)$ is pairwise $T_1$, i.e., singletons are $ij - \theta$-closed. Thus $(X, \tau_1, \tau_2)$ satisfies A.

(b) $\Rightarrow$ (a) If $(X, \tau_1, \tau_2)$ satisfies A, then by Remark 2.5., each singleton is either $ij - \delta$-clopen or $ij - \theta$-closed. Hence $(X, \tau_1^\theta, \tau_2^\theta)$ is pairwise $T_1$ and thus $(X, \tau_1, \tau_2)$ is pairwise Hausdorff.

(b) $\Rightarrow$ (c) Suppose that $(X, \tau_1, \tau_2)$ satisfies A. Then each singleton is clearly either $ji - \delta$-open or $ij - \delta$-closed, i.e., $(X, \tau_1, \tau_2)$ is pairwise almost weakly Hausdorff. If $\{x\}$ is $ij - \delta$-closed, then $\{x\}$ is either $ij - \delta$-clopen or $ij - \theta$-closed, hence always $ij - \theta$-closed.

(c) $\Rightarrow$ (b) This is obvious.

**Theorem 2.8.** For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent:

(a) $(X, \tau_1, \tau_2)$ is pairwise weakly Hausdorff.

(b) $(X, \tau_1, \tau_2)$ satisfies B.
**Proof:** (a) ⇒ (b) If \((X, \tau_1, \tau_2)\) is pairwise weakly Hausdorff, then each singleton is \(ij - \delta -\)closed. Hence \((X, \tau_1, \tau_2)\) satisfies B.

(b) ⇒ (a) This follows from the fact that each \(ij - \theta -\)open singleton must be \(ji -\)clopen.

**Observation 2.9.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(x \in X\). Then

(a) \(\{x\}\) is either \(ij -\)preopen or \(ji -\)nowhere dense (i.e., \(j - Cl(i - Int(\{x\})) = \emptyset\)).

(b) \(\{x\}\) is either \(i -\)open or \(ji -\)preclosed.

(c) \(\{x\}\) is \(i -\)open if and only if \(\{x\}\) is \(ij - \alpha -\)open if and only if \(\{x\}\) is \(ij -\)semi open.

(d) \(\{x\}\) is \(ij -\)preopen if and only if \(\{x\}\) is \(ij - \beta -\)open.

(e) If \(\{x\}\) \(ji -\)nowhere dense, then \(\{x\}\) is \(ij - \alpha -\)closed and thus \(ij -\)semiclosed, \(ij -\)preclosed and \(ij -\)\(\beta -\)closed.

(f) \(\{x\}\) is \(ji -\)semiclosed if and only if \(\{x\}\) \(ji -\)nowhere dense or \(ij -\)regular open.

**3- Characterizations of some separation axioms.**

**Definition 3.1.** A bitopological space \((X, \tau_1, \tau_2)\) is called:

(i) pairwise semi \(T_1\) \([10]\) (resp. pairwise pre \(T_1\), pairwise \(\beta - T_1\) if each singleton is pairwise semiclosed (resp. pairwise preclosed, pairwise \(\beta -\) closed).

(ii) pairwise \(T_{\frac{1}{2}}\) if each singleton is either \(ij - \delta -\)open or \(j -\)closed.

(iii) pairwise semi \(T_{\frac{1}{2}}\) \([10]\) if each singleton is either \(ji -\)semiopen or \(ij -\)semiclosed.

(iv) pairwise feebly \(T_1\) if each singleton is either \(ji -\)nowhere dense or \(ij -\)clopen.

(v) pairwise \(T_{gs}\) if each singleton is either \(ij -\)preopen or \(j -\)closed.

As an immediate consequence of observation 2.9 we note that a bitopological space \((X, \tau_1, \tau_2)\) is pairwise semi \(T_{\frac{1}{2}}\) if and only if each singleton is either \(j -\)open or \(ij -\)\(\alpha -\)closed.

**Proposition 3.2.** For a bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent:

(a) \(X\) is pairwise semi \(T_1\).
(b) each singleton is either $ji - \theta$-open or $ij$ - semiclosed.
(c) each singleton is either $ji - \delta$ - open or $ij$ - semiclosed.
(d) each singleton is either $ji - \delta$ - open or $ij - \alpha$ - closed.

**Proof:** (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Obvious.
(c) $\Rightarrow$ (d): Follows from Observation 2.9.
(d) $\Rightarrow$ (a): Clear.

By observing that a $ji - \theta$ - open set must be $ji$ - clopen and by Observation 2.9 we have a space $(X, \tau_1, \tau_2)$ is pairwise feebly $T_1$ if and only if each singleton is either $ji - \theta$ - open or $ij - \alpha$ - closed. By a similar argument $(X, \tau_1, \tau_2)$ is pairwise pre $T_1$ if and only if each singleton is either $ji - \theta$ - open or $ij$ - preclosed. In addition, $(X, \tau_1, \tau_2)$ is pairwise $T_1$ if and only if each singleton is either $ji - \theta$ - open or $ij$ - closed.

**Proposition 3.3.** For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent:
(a) $X$ is pairwise $\beta - T_1$.
(b) each singleton is either $ji - \theta$ - open or $ij - \beta$ - closed.
(c) each singleton is either $ji - \delta$ - open or $ij - \beta$ - closed.
(d) each singleton is either $ji - \delta$ - open or $ij$ - preclosed.

**Proof:** (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Obvious.
(c) $\Rightarrow$ (d): Let $x \in X$ such that $\{x\}$ is $ij - \beta$ - closed. If $j - \text{Int}(\{x\}) = \emptyset$, then $\{x\} ij$ - preclosed. Otherwise $\{x\}$ is $j$ - open and $ij - \beta$ - closed and so is $ji$ - regular open, i.e., $ji - \delta$ - open.
(d) $\Rightarrow$ (a): Clear.

**Definition 3.4.** A bitopological space $(X, \tau_1, \tau_2)$ is called
(1) pairwise $R_1 [14]$ if for each $x, y \in X$ such that $x \not\in i - \text{Cl}(\{y\})$, there is an $i$ - open set $U$ and a $j$ - open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
(2) pairwise subweakly $T_2$ if $ji - \delta - \text{Cl}(\{x\}) = j - \text{Cl}(\{x\})$ for each $x \in X$.
(3) pairwise pointwise semi regular if each $j$ - closed singleton is $ji - \delta$ - closed.
(4) pairwise pointwise regular if each $j$ - closed singleton is $ji - \theta$ - closed.

**Lemma 3.5.** Let $(X, \tau_1, \tau_2)$ be a bitopological space, then
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Theorem 3.6. For a bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent:

(a) Each singleton is either \(ji - \theta\)-closed or \(ij\)-preopen.

(b) \(X\) is pairwise \(T_{gs}\) and pairwise \(R_1\).

(c) \(X\) is pairwise \(T_{gs}\) and pairwise pointwise regular.

Proof: (a) \(\Rightarrow\) (b): Suppose that each singleton is either \(ji - \theta\)-closed or \(ij\)-preopen. Then \((X, \tau_1, \tau_2)\) clearly is pairwise \(T_{gs}\). Let \(x \in X\). If \(\{x\}\) is \(ij\)-preopen, \(j - Cl(\{x\}) = ij - \theta - Cl(\{x\})\) by Lemma 3.5. If \(\{x\}\) is \(ji - \theta\)-closed then \(\{x\} = ij - \theta - Cl(\{x\}) = j - Cl(\{x\})\). Hence \(X\) is pairwise \(R_1\).

(b) \(\Rightarrow\) (c): Follows immediately from Lemma 3.5.

(c) \(\Rightarrow\) (a): Follows directly from the definitions.

Theorem 3.7. For a bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent:

(a) Each singleton is either \(ji - \delta\)-closed or \(ij\)-preopen.

(b) \(X\) is pairwise \(T_{gs}\) and pairwise subweakly \(T_2\).

(c) \(X\) is pairwise \(T_{gs}\) and pairwise pointwise semi regular.

Proof: (a) \(\Rightarrow\) (b): Suppose that each singleton is either \(ji - \delta\)-closed or \(ij\)-preopen. Then \((X, \tau_1, \tau_2)\) clearly is pairwise \(T_{gs}\). Let \(x \in X\). If \(\{x\}\) is \(ij\)-preopen, then \(j - Cl(\{x\}) = j - Cl(\{x\})\) is \(ji - \theta\)-regular closed set and so \(j - Cl(\{x\}) = ji - \delta - Cl(\{x\})\). If \(\{x\}\) is \(ji - \delta\)-closed, then
obviously we have \( j - \text{CL}(\{x\}) = ji - \delta - \text{CL}(\{x\}) \). Thus \( X \) is pairwise subweakly \( T_2 \).

(b) \( \Rightarrow \) (c) \( \Rightarrow \) (a): It is clear.

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فتتحي هشام خضر
قسم الرياضيات كلية العلوم جامعة أسيوط

في هذا البحث نقدم نوعين جديدين من المجموعات المغلقة المعممة في الفراغات الثنائية التوبولوجية وندرس بعض خواصها وعلاقتها بالأنواع الأخرى من المجموعات المغلقة المعممة في . باستخدام هذه المجموعات نحصل على تشخيص لبعض مسلمات الفصل.