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ON SOME NONSTANDARD DEVELOPMENTS OF INTERMEDIATE VALUE PROPERTY

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In this paper, by using the power of nonstandard analysis tools, we review some of the standard facts on the intermediate value property (IVP) and investigates some new nonstandard developments by extending the classical definition. The notions are generalized to that of any real values; infinitesimals, infinitely close, unlimited. Finally, we give a nonstandard generalization of Sierpinski theorem. We prove that every function can be expressed as a sum of four discontinuous nonstandard functions with infinitesimal intermediate value property (IIVP).

Keyword: IVP, monad, galaxy, continuity, s-continuity, internal functions, Sierpinski

INTRODUCTION

In this paper, we use E. Nelson's nonstandard analysis construction [10], based on the theory which is called internal set theory and denoted by IST. The axioms of IST are those axioms of Zermelo-Frankel with the axiom of choice (ZFC), together with three new axioms (Principles) which are; Transfer, Idealization, and Standardization. Every set defined in ZFC is *standard*, every mathematical objects: a real number, function, … , etc in ZFC is regarded to be a set, and any set or formula in IST is called *internal* in case it does not defended with the new predicate "standard" and its derivations, otherwise it is called *external*. Among the three principles, we used here, the transfer principle (TP) which ensures that, if $F(x)$ is a standard statement, $F(x)$ holds if and only if it holds for all standard x. More generally;

If $F(x, t_1, t_2, \dots, t_n)$ is an internal formula with free variables $x, t_1, t_2, \dots, t_n$, then

$$\forall^{st} t_1, t_2, \dots, t_n (\forall^{st} x\, A(x, t_1, t_2, \dots, t_n) \Longrightarrow \forall x\, A(x, t_1, t_2, \dots, t_n)).$$

A real number x is called *limited* if $|x| \leq r$ for some positive standard real numbers r, *unlimited* if $|x| > r$ for all positive standard real numbers r, *infinitesimal* if $|x| < r$ for all positive standard real numbers r, *appreciable* in case x is limited not infinitesimal. Two real numbers x and y are said to be *infinitely near* or *infinitely close* if $x - y$ is infinitesimal and denoted by $x \simeq y$. The set of all real numbers y for which $x - y$ is limited, is called galaxy of x and denoted by *gal* (x). Let x be a real number. Then the set of all real numbers which are infinitely close to x is called the *monad* or *halo* of x and denoted by $m\ (x)$. Let A be a subset of a space E. The set of all points $y \in E$ for which there exist $x \in A$ such that$x \simeq y$, is called the monad of A and denoted by $m\ (A)$. Let x be a limited real number. Then it is infinitely close to a unique standard real number. This real number is called *standard part* or *shadow* of x and denoted by $st(x)$, $sh\ (x)$ or $°x$. The *shadow* of a set A, denoted by $sh(A)$ or $°A$, is the unique standard set whose standard elements are precisely those whose monad intersects is A. Let $f : A \to B$ be a function. We call f; an *internal* when it is internal as a relation, *continuous* at x_o if and only if f and x_o are standards and $f(x) \simeq f(x_o)$ for all $x \simeq x_o$, *s-continuous* if and only if $f(x) \simeq f(x_o)$ for all $x \simeq x_o$. A standard sequence $\{x_n\}$ is *converges to* x if and only if $x_n \simeq x$ for all unlimited n. Let $\boldsymbol{E}$ be a subset of $\mathbb{R}$. If p is a limit point of $\boldsymbol{E}$, then there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\boldsymbol{E}$ with $p_n \neq p$ for all $n \in \mathbb{N}$, such that $p_n \simeq p$ for all unlimited n. If $\{x_n\}$ is a sequence such that $x_n \simeq 0$ for all standard *n*, there exists a positive unlimited integer ω such that $x_n \simeq 0$ for all $n \leq \omega$ (*Robinson Lemma*). Throughout this paper, the following symbols will be used:

- The symbol $\gtrsim$ represents the external relation *greater than or infinitesimal close*.
- The symbol $\succ\simeq$ represents the external relation *greater than and infinitely close*.
- The symbol $m^+(x)$ represents the external set *right half of the monad of* x.
- The symbol $gal^+(x)$ represents the external set *right half of the galaxy of* x.

Similarly, we can use the representation of the symbols;

$$\lesssim, \prec\simeq, \lnsim, \gnsim, m^-(x), gal^-(x).$$

For the above definitions and other nonstandard concepts see [5-7, 10, 12].

Definition 1.1 (*Intermediate Value Property-IVP*)[1]
A function $f: \mathbb{R} \to \mathbb{R}$ is said to have the *intermediate value property* (**IVP**) provided that if a and b are real numbers such that $a < b$ and $f(a) \neq f(b)$, then for every λ between $f(a)$ and $f(b)$, there exists a real number z; $a < z < b$ such that $f(z) = \lambda$. That is, the image of every interval is an interval.

In 1875 G. Darboux showed that there exist functions with the intermediate value property that are not continuous [4]. Because of his work with functions having the intermediate value property, these functions are called Darboux functions. Some classical works about properties of functions with IVP can be found in [1-3, 9, 11, 13]. In this article we construct some unusual and unintuitive functions which have interesting properties. In the first part we will give an equivalent nonstandard definition of *Intermediate Value Property*, and then generalize the notion to the extendable region of nonstandard values to contain (infinitesimals, infinitely close, unlimited) real numbers. Finally in the last part, we generalize the famous Sierpinski theorem to the sum of four nonstandard functions.

MAIN RESULTS

By using (TP), an equivalent statement of Definition 1.1 in nonstandard sense can be obtained as follows:

Definition 2.1 (*Intermediate Value Property- Standard Version*,(*IVP-SV*))
A standard function $f: \mathbb{R} \to \mathbb{R}$ is said to have the *intermediate value property* provided that if a and b are standard real numbers such that $a < b$ and $f(a) \neq f(b)$, then for every standard λ between $f(a)$ and $f(b)$, there exists a standard real number z; $a < z < b$ such that $f(z) = \lambda$.

The following definition is a nonstandard generalization of Definition 2.1 such that the domain includes nonstandard values, and results include nonstandard outcomes.

Definition 2.2 (*Infinitesimal Intermediate Value Property*, (IIVP))
An internal function $f: \mathbb{R} \to \mathbb{R}$ is said to have the *infinitesimal intermediate value property* provided that if a and b are real numbers such that $a < b$ and $f(a) \neq f(b)$, then for every λ between $f(a)$ and $f(b)$, $f(a) \lesssim \lambda \lesssim f(b)$, there exists a real number z ; $x \lesssim z \lesssim y$ such that $f(z) \simeq \lambda$.

Unfortunately, Definition 2.1 hold only for standards λ and f. Here, the nonstandardist is caught in a dilemma after where using classical tools, between what appears attractive and what is demanded by reality. There are several problems which fails to hold for standard values of λ, although the problem still coherent and practically not break down if there exist some $z \in m((a,b))$ such that $f(z) \simeq \lambda$ for $\lambda \in m((f(a), f(b)))$. So, we will see that the nonstandard tools make it possible to reformulate, in an often simpler setting, the basic definition of **IVP**. We will observe that the new characterization given by Definition 2.2 hold for standard and nonstanderd entities. Definition 2.2 it is stronger than both Definitions 1.1 and 2.1 because equality implies the infinitesimally close and internality of the function f includes the case where f is standard or nonstandard.

Lemma 2.3.
Let $f\colon \mathbb{R} \to \mathbb{R}$. Then f satisfies IVP if and only if it is satisfies the IVP-S V.

Proof:
Applying backward and forward direction of (**TP**) to Definitions 1.1 and 2.1 respectively we get the result. ■

In standard analysis it is known that every continuous function has IVP. The same argument is also true in nonstandard analysis for both continuity and s-continuity. Furthermore, IVP implies IIVP but the converse is not hold in general, as shown in the following example.

Example 2.4.
Let f:[0, ε] $\to \mathbb{R}$ defined by $f(x) = e^{\frac{x}{\varepsilon}}$ where ε is infinitesimal. Then f satisfies IIVP yet f is not s- continuous at $x = \varepsilon$. On the other hand, the example shows that for any standard value λ; $f(0) < \lambda < f(\varepsilon)$ there is no standard value z; $0 < z < \varepsilon$ such that $f(z) = \lambda$. This means that IIVP does not imply IVP.

Lemma 2.5.
Let $f\colon \mathbb{R} \to \mathbb{R}$ and g be a function such that $f(x) \simeq g(x)$ for all $x \in \mathbb{R}$. If f satisfies IIVP, then so does g.

***Proof*:**
Assume that f satisfies IIVP and $f(x) \simeq g(x)$ for all $x \in \mathbb{R}$. Then for every λ such that $f(a) \lesssim \lambda \lesssim f(b)$, where $a\,, b \in \mathbb{R}$, there exists z;$a \lesssim z \lesssim b$ such that $f(z) \simeq \lambda$.
Now, to show that g satisfies IIVP on $\mathbb{R}$. Let γ be a real number strictly between $g(a)$ and $g(b)$, for $a, b \in \mathbb{R}$, to prove that there exists β; $a \lesssim \beta \lesssim b$ such that $g(\beta) \simeq \gamma$.
Since $f(x) \simeq g(x)$ for all $x \in \mathbb{R}$, then we have the following cases:
1) $f(a) \prec\simeq g(a) \lesssim \gamma \lesssim g(b) \prec\simeq f(b)$.
2) $f(a) \prec\simeq g(a) \lesssim \gamma \lesssim f(b) \prec\simeq g(b)$.
3) $g(a) \prec\simeq f(a) \lesssim \gamma \lesssim g(b) \prec\simeq f(b)$.
4) $g(a) \prec\simeq f(a) \lesssim \gamma \lesssim f(b) \prec\simeq g(b)$.
5) $g(a) \lesssim \gamma \prec\simeq f(a) \lesssim f(b) \prec\simeq g(b)$.
6) $g(a) \prec\simeq f(a) \lesssim f(b) \lesssim \gamma \prec\simeq g(b)$.

In the above first four cases we have $f(a) \lesssim \gamma \lesssim f(b)$ and from being f satisfies IIVP then there existsβ; $a \lesssim \beta \lesssim b$ such that $f(\beta) \simeq \gamma$ but $g(\beta) \simeq f(\beta)$, therefore $g(\beta) \simeq \gamma$. For the case (5), since $f(x) \simeq g(x)$ for all $x \in \mathbb{R}$, it follows that $\gamma \simeq g(a) \simeq f(a)$. According to Definition 2.2, it is sufficient to take $z = a$. The proof of (6) is similar. ■

Lemma 2.6.

Let f:[a,b]→ $\mathbb{R}$ be a function satisfies IIVP with the same hypotheses of Definition 2.2 and let $z \simeq u$:

1. *If f is s-continuous, then $f(u) \simeq \lambda$.*
2. *If f is limited continuous, then $^{o}f(u) \simeq \lambda$.*

Proof:

1. Assume that z $\simeq u$. Since f is s-continuous, then $f(z) \simeq f(u)$. We conclude from IIVP of f that $f(z) \simeq \lambda$, hence $f(u) \simeq \lambda$.

2. Assume that f is limited continuous. Since f is continuous, then f is standard and by the definition of *standard part* we have, for all z ∈ ℝ, $f(z)$ is infinitely close to a unique standard real number. Hence $f(z) \simeq {}^{\circ}f(u)$. Therefore by the first part we deduce ${}^{\circ}f(u) \simeq \lambda$. ■

Lemma 2.7.

Let $f: \mathbb{R} \to \mathbb{R}$ be s-continuous. If $a, b \in \mathbb{R}$ such that $f(a) \simeq f(a+b)$, then for all standard positive integer k there exists β; $a \lesssim \beta \lesssim b$ such that $f\left(\beta + \frac{b}{k}\right) \simeq f(\beta)$.

Proof:

Let k be standard positive integer. Define $g: [a, b + \frac{b}{k}] \to \mathbb{R}$ by

$$g(x) = f\left(x + \frac{b}{k}\right) - f(x) \qquad \ldots (2.1)$$

Since f is s-continuous, then so is g. Hence g has IIVP. If we prove that there exist $\beta \in [a, b + \frac{b}{k}]$ such that $g(\beta) \simeq 0$, then the assertion follows. Now, for $x \in \left[a, b + \frac{b}{k}\right]$ let $g(x) \gtrsim 0$. From (2.1) we conclude that

$$f(a) \lnapprox f\left(a + \frac{b}{k}\right) \lnapprox f\left(a + \frac{2b}{k}\right) \lnapprox \cdots \lnapprox f(a+b) \qquad \ldots (2.2)$$

Which is impossible because $f(a) \simeq f(a+b)$. the impossibility of the case $g(x) \lnapprox 0$ may be proved in similar way. Hence there exist $\beta \in: [a, b + \frac{b}{k}]$ such that $g(\beta) \simeq 0$. As a consequence of the last result, we get $f\left(\beta + \frac{b}{k}\right) \simeq f(\beta)$. ■

In recent years a number of articles have dealt with questions concerning the IVP of the sum of two real functions. Ciesielski and Pawlikowski [3](see also [1,8]) showed that for every Darboux function $g: \mathbb{R} \rightarrow \mathbb{R}$ there exists a continuous nowhere constant function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f + g$ is Darboux. Classically known that the sum of a non-constant continuous function and a function has IVP might fail to have IVP. In the following theorem we study the case where f and g are nonstandard.

Theorem 2.8.

Let f and g be two limited infinitely close functions. If f and g are satisfying IIVP, then so is $f + g$.

Proof **:**

Let λ be such that $(f+g)(x) \lesssim \lambda \lesssim (f+g)(y)$. To find z ; $x \lesssim z \lesssim y$ such that $(f+g)(z) \simeq \lambda$. We can prove the requirement according to the following two cases of limitedness of both f and g:

*Case I: If both **f** and **g** are infinitesimal.*

Let $(f+g)(x) \lesssim \lambda \lesssim (f+g)(y)$.
Since both f and g are infinitesimals, then so is $f + g$.
Thus $(f+g)(x) \simeq 0$ for all x. Therefore $\lambda \simeq 0$.
Hence $(f+g)(z) \simeq \lambda$ for all z ; $x \lesssim z \lesssim y$.

*Case II: If both **f** and **g** are appreciable*

Since both f and g are infinitesimally close, it follows that
$(f+g)(x) = f(x) + g(x) \simeq 2f(x) \simeq 2g(x)$ for all x. Therefore,

$$2f(x) \simeq f(x) + g(x) = (f+g)(x) \lesssim \lambda \lesssim (f+g)(y)$$
$$= f(y) + g(y) \simeq 2f(y)$$

That is $f(x) \lesssim \frac{\lambda}{2} \lesssim f(y)$.

Since f satisfies IIVP, then there exists z between x, y such that $f(x) \simeq \frac{\lambda}{2}$. Thus $(f+g)(z) \simeq 2f(z) \simeq \lambda$.

The same proof can be obtained when we drop the assumption $(f+g)(x) \simeq 2g(x)$, which completes the proof.■

Lemma 2.9.

Let f_n be a sequence of functions satisfies IIVP for all n. Suppose that f_n converges uniformly to f. Then f satisfies IIVP.

***Proof*:**

Let λ be such that $f(x) \lesssim \lambda \lesssim f(y)$. Since f_n is converges uniformly to f, then $f_n(t) \simeq f(t)$ for all unlimited n and for all t, standard or not. To prove the requirement, it suffices to fix an unlimited n where $f_n(x) \simeq f(x)$ then the proof is consequence of Lemma 2.5. That is there exists $z; x \lesssim z \lesssim y$ such that $f(z) \simeq \lambda$. ■

It is easy to deduce from Lemma 2.9 that for a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of values corresponding with the values of the sequence f_n; $f_n(x) \lesssim \lambda_n \lesssim f_n(y)$ for all $n \in \mathbb{N}$, there exists $z; x \lesssim z \lesssim y$ such that λ_n converges uniformly to $f(z)$.

Lemma 2.10.

If f and g are satisfying IIVP and g is s-continuous, then fog it satisfies IIVP.

Proof:

Let λ be such that $g(f(x)) \lesssim \lambda \lesssim g(f(y))$. To find z ; $x \lesssim z \lesssim y$ such that $g(f(z)) \simeq \lambda$. Since g satisfies IIVP, then for λ; $g(f(x)) \lesssim \lambda \lesssim g(f(y))$ there exists w; $f(x) \lesssim w \lesssim f(y)$ such that $g(w) \simeq \lambda$.
Now, since f satisfies IIVP, then for $w; f(x) \lesssim w \lesssim f(y)$ there exists z; $x \lesssim z \lesssim y$ such that $f(z) \simeq w$. Since g is s-continuous, then $g\big(f(z)\big) = g(w)$. Thus $\lambda \simeq g(w) \simeq g(f(z))$. ■

Lemma 2.11.

Let X be a connected space and let $f: X \to \mathbb{R}$ be a continuous function. Suppose that g is defined from $f(X)$ into $\mathbb{R}$. If fog is continuous, then g satisfies IIVP.

Proof:

Let $g(x) \lesssim \lambda \lesssim g(y)$ for some λ. We must find z; $x \lesssim z \lesssim y$ such that $g(z) \simeq \lambda$. Let $\varphi = fog$ and let $G = \{(f(t), \varphi(t)) \in \mathbb{R}^2, t \in X\}$. G is the graph of g.
Since f and φ are continuous then the map $t \longmapsto (f(t), \varphi(t))$ is continuous and G is connected. Let

$$V = \{[(z,b) \in G: z < x)] \vee [x \lesssim z \lesssim b \lesssim \lambda]\}, \text{ and}$$
$$W = \{[(z,c) \in G: z > y)] \vee [x \lesssim z \lesssim \lambda \lesssim c]\}.$$

Then V and W are closed subsets of G with $G = V \cup W$.
Thus $\big(x, g(x)\big) \in V$ if and only if $V \neq \emptyset$, and $\big(y, g(y)\big) \in W$ if and only if $W \neq \emptyset$. By connectedness of G, $V \cap W \neq \emptyset$. Say $(z, \lambda^*) \in V \cap W$.

It follows that $\lambda^* \simeq \lambda$ and that $x \lesssim z \lesssim y$. So $g(z) \simeq \lambda$, and the IIVP has been established. ■

Theorem 2.12.
Let $f:[a,b] \to \mathbb{R}$ be a function satisfies IIVP. If for each $y \in \mathbb{R}$, the set $\{x \in [a,b]: f(x) \simeq y\}$ is closed, then f is s-continuous on $[a,b]$.

Proof:
Let $t \in [a,b]$. By hypothesis the set

$E = \{x \in [a,b]: f(x) <\simeq f(t)\} \cup \{x \in [a,b]: f(x) >\simeq f(t)\}$ is closed. Since $t \notin E$, there exists $m(t)$ such that $m(t) \cap E = \emptyset$.

To prove that f is s-continuous on $[a,b]$ we need to show that $f(x) \in m(f(t))$ for all $x \in m(t)$.
By contradiction, suppose that $f(x) \notin m(f(t))$ for some $x \in m(t)$ with $x \in m^+(t)$. That is $f(x) \not\simeq f(t)$. Thus there exist a standard ε such that $|f(x) - f(t)| \geq \varepsilon$. Hence $f(x) \geq f(t) + \varepsilon$ or $f(x) \leq f(t) - \varepsilon$.
Consider the case where $f(x) \leq f(t) - \varepsilon$, then $f(x) \leq f(t) - \varepsilon \leq f(t)$.
Since f satisfies IIVP, there exist $z\,;\, t < z < x$ such that $f(z) \simeq f(t) - \varepsilon$.
But this is not possible since $m(t) \cap E = \emptyset$. The impossibility of the other case can be proved in a similar way.
Hence, f is s-continuous at t. ■

NONSTANDARD GENERALIZATION OF SIERPINSKI THEOREM

For x, y in $\mathbb{R}$, define the relation $\sim$ by : $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. The relation $\sim$ has the following properties:

1. Is an equivalence relation.
2. For any $x \in \mathbb{R}$, $[x] = \{y \in \mathbb{R}: y \sim x\}$ is the equivalence class contains x.
3. $\mathbb{R} = \bigcup_{x \in R} [x]$ and if $y \notin [x] \Longrightarrow [x] \cap [y] = \emptyset$.

Let $E = \{[x]: x \in \mathbb{R}\}$. Define $\mu: \mathbb{R} \to E$, by $\mu(x) = [x], \forall x \in \mathbb{R}$ which is onto and $card\ E = card\ \mathbb{R}$

Theorem 3.1.
There exists an appreciable nowhere discontinuous function $f: \mathbb{R} \to \mathbb{R}$ satisfies IIVP and map any monad onto galaxy.

***Proof*:**
Let $t \in \mathbb{R}$. Define $G^{+}(t) = \{y \in \mathbb{R}: y - t \ \ is\, positive\, limited\}$. It is clear that $cardG^{+} = card\ E$, so we can find a bijective function g: $E \rightarrow G^{+}$. Define f: $\mathbb{R} \rightarrow G^{+}$ by $f(x) = g([x])$.
We shall now prove that for any $t \in \mathbb{R}$ the image of $m(t)$ through f is exactly $G^{+}(t)$. We notice that $[x] = x + \mathbb{Q} = \{x + q: q \in \mathbb{Q}\}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, any of its translations is also dense in $\mathbb{R}$, meaning that any equivalence class from E has at least one element common with any $m(t)$.
Thus

$$f\big(m(t)\big) = g(E) = G^{+}(t) \qquad \cdots (3.1)$$

It remains to prove that f satisfies IIVP, for this purpose, we shall show that the image of every interval is an interval. From definition of $G^{+}(t)$, we conclude that

$$G^{+}(t) = \bigcup_{l\ is\ limited} [t, l], \qquad \cdots (3.2)$$

which is a closed interval.

Now, $$f([a, b]) = \bigcup_{t \in [a,b]} \ \bigcup_{l\ is\ limited} [t, l].$$

The consequence of this last result, is also interval. In order to prove that f is not continuous, take any infinitesimal interval $(t - \delta, t + \delta)$, $t \in \mathbb{R}$ and $\delta \simeq 0$. From (3.1) and (3.2), we conclude that $f((t - \delta, t + \delta))$=$g(E)$= $[t, l]$ for some limited l. Hence f is discontinuous at t. ∎

Corollary 3.2.
There exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which satisfies IIVP and takes any real value in any monad or galaxy of any point in $\mathbb{R}$.

Proof:
We proceed the same way of the above proof, defining the bijection $g: E \rightarrow \mathbb{R}$ and the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = g([x])$. We see that $f((s, t))$= $\mathbb{R}$ for any interval (s, t), using a similar argument of the above theorem we get $f(\mathbb{I})$= $\mathbb{R}$, for any interval $\mathbb{I}$, hence f satisfies IIVP. ∎

In the following theorem, we give a nonstandard generalization of Sierpiñski Theorem [9, 13] about decomposing any function into two discontinuous functions satisfies IVP. The extension leads to decomposing any function into four discontinuous nonstandard functions with IIVP.

Theorem 3.3.
For any function $f: \mathbb{R} \rightarrow \mathbb{R}$ there exist functions $f_1, f_2, f_3, f_4: \mathbb{R} \rightarrow \mathbb{R}$ satisfies IIVP and are discontinuous at any point in $\mathbb{R}$, such that $f = f_1 + f_2 + f_3 + f_4$.

Proof:
Take $f: \mathbb{R} \to \mathbb{R}$. Consider a bijection $g: E \to \mathbb{R}$ and we denote $\mathcal{A}_1 = gal^-(0)$, $\mathcal{A}_2 = gal^+(0)$, $\mathcal{A}_3 = m(0)$, and $\mathcal{A}_4 = \mathbb{R} \backslash (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$. Thus

$$card\mathcal{A}_1 = card\mathcal{A}_2 = card\,\mathbb{R},$$

and we can define the bijections $g_1: \mathbb{R} \to \mathcal{A}_1$, $g_2: \mathbb{R} \to \mathcal{A}_2$, $g_3: \mathbb{R} \to g(\mathcal{A}_3)$, and $g_4: \mathbb{R} \to g(\mathcal{A}_4)$. Now, define

$$f_1(t) = \begin{cases} f(t) - x & t \in g_1(x) \\ x - \varepsilon & t \in g_2(x) \\ \varepsilon - x & t \in g_3(x) \\ x & t \in g_4(x) \end{cases}, \qquad f_2(t) = \begin{cases} x & t \in g_1(x) \\ f(t) - x & t \in g_2(x) \\ x - \varepsilon & t \in g_3(x) \\ \varepsilon - x & t \in g_4(x) \end{cases},$$

$$f_3(t) = \begin{cases} \varepsilon - x & t \in g_1(x) \\ x & t \in g_2(x) \\ f(t) - x & t \in g_3(x) \\ x - \varepsilon & t \in g_4(x) \end{cases}, \qquad f_4(t) = \begin{cases} x - \varepsilon & t \in g_1(x) \\ \varepsilon - x & t \in g_2(x) \\ x & t \in g_3(x) \\ f(t) - x & t \in g_4(x) \end{cases}.$$

Where ε is an infinitesimal. It is clear from their definition that $f = f_1 + f_2 + f_3 + f_4$. Let us prove that f_i, $i = 1,2,3,4$ have the IIVP. Consider $\mathbb{I} \subset \mathbb{R}$ an interval. Since any equivalence class in *E* is dense in $\mathbb{R}$, we have

$S_1 = \mathbb{I} \cap (g_1(x) \cup g_4(x)) \neq \emptyset$ $\qquad$ $S_2 = \mathbb{I} \cap (g_2(x) \cup g_4(x)) \neq \emptyset$
$S_3 = \mathbb{I} \cap (g_1(x) \cup g_3(x)) \neq \emptyset$ $\qquad$ $S_4 = \mathbb{I} \cap (g_2(x) \cup g_3(x)) \neq \emptyset$

Therefore from definitions of f_i, $i = 1,2,3,4$ we see that
$x \in f_1(S_2 \cup S_4) \subset f(\mathbb{I})$, $\qquad$ $x \in f_2(S_1 \cup S_3) \subset f(\mathbb{I})$,
$x \in f_3(S_1 \cup S_2) \subset f(\mathbb{I})$, $\qquad$ $x \in f_4(S_3 \cup S_4) \subset f(\mathbb{I})$.
For all $x \in \mathbb{R}$, yielding $f_1(\mathbb{I}) = f_2(\mathbb{I}) = f_3(\mathbb{I}) = f_4(\mathbb{I}) = \mathbb{R}$. We conclude that f_i, $i = 1,2,3,4$ satisfies IIVP. We see that $f_1(\mathbb{I}) = f_2(\mathbb{I}) = f_3(\mathbb{I}) = f_4(\mathbb{I}) = \mathbb{R}$ for any interval $\mathbb{I}$ so the function $f_i, i = 1,2,3,4$ are not continuous at any points $x_o \in \mathbb{R}$, because
$f(m(0))=\mathbb{R} \not\subset m(f(0)) = Gal(0)$. ■

REFERENCES

1. A. Bruckner, J. Ceder, On the sums of Darboux functions, Proc. Amer. Math. Soc., 51, 97–102 (1975).
2. K. Banaszewski, Algebraic properties of functions with the Cantor intermediate value property, Math. Slovaca, 48(2), 173-185 (1998).
3. K. Ciesielski, J. Pawlkowski, On Sums of Darboux and Nowhere Constant Continuous Functions, P. Am. Math. Soc., 130(7), 2007-2013 (2001).
4. G. Darboux, Memoire sur les fonctions discontinues. Ann. Sci. Scuola. Norm. Sup. 4, 57-112 (1875).
5. F. Diener, M. Diener, Nonstandard Analysis in Practice, Springer-Verlag Berlin Heidelberg (1996).
6. A.E. Hurd, P.A. Loeb, An Introduction to Nonstandard Real Analysis, Academic Press, Inc. (1985).
7. A.G. Kusraev, S.S. Kutateladze, Nonstandard Methods of Analysis, Kluwer Academic Publishers (1994).
8. A. Maliszewski, Maximums of Darboux Baire one functions, Math. Slovaca. 56(4), 427-431 (2006).
9. S. Marcus, Sur la représentation d'une fonction arbitraire par des fonctions jouissant de la propriété de Darboux, Trans. Amer. Math. Soc. 95(3), 489-494 (1960).
10. E. Nelson, Internal set theory, a new approach to nonstandard analysis, Bull. Amer. Math. Soc. 83(6), 1165–1198 (1977).
11. D.A. Neuser, S.G. Wayment, A Note on the Intermediate Value Property, Am. Math. Mon. 81(9), 995-997 (1974).
12. A. Robinson, Nonstandard Analysis, rev. ed.. Princeton University Press (1996).
13. W. Sierpiński, sur une propriété de fonctions réelles quelconques, Matematiche Catania. 8(2), 43–48 (1953).

حول بعض التطويرات غير القياسية لخاصية القيمة الوسطى

في هذا البحث، استخدمنا قوة أدوات التحليل غير القياسي وقد تم مراجعة بعض الحقائق و النتائج القياسية حول خاصية القيمة الوسطى وايجاد بعض النتائج غير القياسية جديدة وذلك بتعميم المفاهيم القياسية الموجودة لتشمل اية قيمة حقيقية (قياسية وغير قياسية) على سبيل مثال قيم غير قياسية غير متناهية في الصغر وغير متناهية في الكبر وغير متناهية القرب. و تم اعطاء تعميم غير قياسي للنظرية سربنسكي حول تمثيل دالة معطاة كمجموع اربع دوال غير قياسية و غير مستمرة مع خاصية نظرية ($IIVP$).