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GENERALIZED $\delta-$ SEMICLOSED SETS IN BITOPOLOGICAL SPACES

F. H. Khedr

Department of Mathematics-Faculty of Sciences-University of Assiut- Assiut 715161- Egypt
E.mail : Khedrfathi@gmail.com

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In this paper we introduce and study $ij-g\delta s-$closed sets and study some of its properties and its relations with other kinds of generalized closed sets in bitopological spaces. Using these sets we obtain a characterization of pairwise $T_{\frac{3}{4}}$ spaces. Also we introduce the concepts of pairwise $g\delta s-$continuous and pairwise $g\delta s-$irresolute functions. Finally we define the concept of pairwise $g\delta sc-$homeomorphism and prove that the set of all pairwise $g\delta sc-$homeomorphisms from (X,τ_1,τ_2) onto itself has a group structure under the composition.

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INTRODUCTION

Throughout this paper, (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or simply *X* and *Y*) always mean bitopological spaces (or simply spaces) on which no separation axioms are assumed unless explicitly stated. Also, $i,j=1,2$ and $i\neq j$. Let *A* be a subset of a space (X,τ_1,τ_2). The closure of *A* and the interior of *A* in the topological space (X,τ_i) are denoted by $i-Cl(A)$ and $i-Int(A)$, respectively. We write $i-$ open (resp. $i-$closed) set to mean that the set is open (resp. closed) in the topological space (X,τ_i). A subset *A* of a space (X,τ_1,τ_2) is said to be $ij-$ regular open (resp. $ij-$regular closed) [12] if $A=i-Int(j-Cl(A))$ (resp. $A=i-Cl(j-\text{int}(A))$). The collection of all $ij-$regular open sets form a base for a topology τ_i^* is coarser than τ_i [10]. The bitoplogical space (X, τ_1^*, τ_2^*) is called the semi-regularization of (X, τ_1, τ_2). If $\tau_i^* = \tau_i$, then (X, τ_1, τ_2) is said to be pairwise semi regular. The $ij-\delta-$ interior [10] of a subset A of a space X is the union of all $ij-$ regular open sets of X contained in A and is denoted by $ij-\delta-Int(A)$. The subset A of X is called $ij-\delta-$ open if $A = ij-\delta-Int(A)$,i.e., a set is $ij-\delta-$open if it is the union of $ij-$ regular open sets. The complement of an $ij-\delta-$open set is called $ij-\delta-$ closed. A point $p \in X$ is in the $ij-\delta-$ closure of A [1] if $i-Int(j-Cl(U)) \cap A \neq \phi$ for every $U \in \tau_i$ and $p \in U$. The set of all $ij-\delta-$ closure points of A is denoted by $ij-\delta-Cl(A)$. Obviously A is $ij-\delta-$ closed if and only if $A = ij-\delta-Cl(A)$. The family of all $ij-\delta-$ open sets forms a topology on X denoted by $\tau_{i\delta}$ [10]. It is well known that $\tau_i^* = \tau_{i\delta}$ [10]. A subset A of X is called $ij-$ semiopen [2] (resp. $ij-\alpha-$ open [5], $ij-$ preopen [5]) if $A \subset j-Cl(i-Int(A))$(resp. $A \subset i-Int(j-Cl(i-Int(A)))$, $A \subset i-Int(j-Cl(A))$). The complement of a $ij-$ semiopen (resp. $ij-\alpha-$ open, $ij-$ preopen) set is called $ij-$ semiclosed (resp. $ij-\alpha-$closed, $ij-$ preclosed).

In this paper we introduce and study $ij-g\delta s-$closed sets and study some of its properties and its relations with other kinds of generalized closed sets in bitopological spaces. Using these sets we obtain a characterization of pairwise $T_{\frac{3}{4}}$ spaces. Also we introduce the concepts of pairwise $g\delta s-$continuous and pairwise $g\delta s-$irresolute functions. Finally we define the concept of pairwise $g\delta sc-$ homeomorphism and prove that the set of all pairwise $g\delta sc-$homeomorphisms from (X, τ_1, τ_2) onto itself has a group structure under the composition.

2. $ij-\delta-$Semi open sets.

Definition 2.1. A subset A of bitopological space (X, τ_1, τ_2) is called $ij-\delta-$ semi open if there exists an $ij-\delta-$open set U such that $U \subset A \subset j-Cl(U)$. The complement of a $ij-\delta-$semi open set is called $ij-\delta-$semiclosed.

A point $x \in X$ is called a $ij-\delta-$semi cluster point of *A* if $A \cap U \neq \phi$ for every $ij-\delta-$semi open set *U* of *X* containing *x*. The set of all $ij-\delta-$semi cluster points of *A* is called the $ij-\delta-$semi closure of *A* and is denoted by $ij-\delta-sCl(A)$. The collection of all $ij-\delta-$semi open (resp. $ij-\delta-$semiclosed) sets of *X* will be denoted by $ij-\delta SO(X)$ (resp. $ij-\delta SC(X)$).

A subset *U* of a space *X* is called $ij-\delta-$semi neighborhood (briefly, $ij-\delta$-semi nbd) of a point *x* in *X* if there exists a $ij-\delta-$semi open set *V* such that $x \in V \subseteq U$.

Lemma 2.2. The union of arbitrary collection of $ij-\delta-$semi open sets in (X,τ_1,τ_2) is $ij-\delta-$semi open.

Proof: Since arbitrary union of $ij-\delta-$open sets is $ij-\delta-$open [6 Lemma 2.2], the result follows directly.

Lemma 2.3. The intersection of arbitrary collection of $ij-\delta-$semiclosed sets in (X,τ_1,τ_2) is $ij-\delta-$semiclosed.
Proof: Follows directly by Lemma 2.1.

Corollary 2.4. For a subset *A* of a bitopological space (X,τ_1,τ_2), we have $ij-\delta sCl(A) = \cap\{F : A \subseteq F,\ F \in ij-\delta SC(X)\}$.

Corollary 2.5 For a subset *A* of a bitopological space (X,τ_1,τ_2), we have $ij-\delta sCl(A)$ is $ij-\delta-$semiclosed, that is $ij-\delta sCl(ij-\delta sCl(A)) = ij-\delta sCl(A)$.

Lemma 2.6. For subsets *A*, *B* and $A_k\,(k \in I)$ of a bitopological space (X,τ_1,τ_2), the following hold

(1) $A \subseteq ij-\delta sCl(A)$.
(2) $A \subseteq B \Rightarrow ij-\delta sCl(A) \subseteq ij-\delta sCl(B)$.
(3) $ij-\delta sCl(\bigcap_k A_k) \subseteq \bigcap_k \{ij-\delta sCl(A_k)\}$.
(4) $ij-\delta sCl(\bigcup_k A_k) = \bigcup_k \{ij-\delta sCl(A_k)\}$.
(5) *A* is $ij-\delta-$semiclosed if and only if $A = ij-\delta sCl(A)$.

Recall that a subset *A* of a bitopological space (X,τ_1,τ_2) is called $ij-g-$closed [3] (resp. $ij-gs-$ closed [8]) if $j-Cl(A) \subset U$ (resp.

$ji-sCl(A)\subset U$) whenever $A\subset U$ and U is $i-$ open in X. Now, we introduce the following concepts.

Definition 2.7. A subset A of a bitopological space (X,τ_1,τ_2) is called:

(1) $ij-\delta gs-$ closed, if $ji-\delta-sCl(A)\subset U$ whenever $A\subset U$ and U is $ij-\delta-$ open in X.

(2) $ij-\delta g-$ closed, if $ji-\delta-Cl(A)\subset U$ whenever $A\subset U$ and U is $i-$ open in X.

(3) $ij-g\delta-$ closed, if $j-Cl(A)\subset U$ whenever $A\subset U$ and U is $ij-\delta-$ open in X.

(4) $ij-\delta g^*-$ closed, if $ji-\delta-Cl(A)\subset U$ whenever $A\subset U$ and U is $ij-\delta-$ open in X.

The complement of a $ij-g-$ closed (resp. $ij-gs-$ closed, $ij-\delta gs-$ closed) set is called $ij-g-$ open [3] (resp. $ij-gs-$ open [8], $ij-\delta gs-$ open).

3- $ij-g\delta s-$ closed sets

Definition 3.1. A subset A of a bitopological space (X,τ_1,τ_2) is said to be $ij-g\delta s-$ closed if $ji-\delta-sCl(A)\subset U$ whenever $A\subset U$ and U is $i-$ open set in X.

Remark 3.2. For a subset of a bitopological space, from definition we have the following diagram of implications

$$\begin{array}{ccccccc}
ji-\delta-\text{ closed} & \rightarrow & ij-\delta g-\text{ closed} & \rightarrow & ij-\delta g^*-\text{ closed} & \rightarrow & ij-g\delta-\text{ closed} \\
\downarrow & & \downarrow & & \downarrow & & \\
ji-\delta-\text{ semiclosd} & \rightarrow & ij-g\delta s-\text{ closed} & \rightarrow & ij-\delta gs-\text{ closed} & &
\end{array}$$

Where none of these implications is reversible.

Definition 3.3. A bitoplogical space (X,τ_1,τ_2) is called a pairwise partition space if every $i-$ open set is $j-$ closed.

Theorem 3.4. For a subset A of a partition space (X,τ_1,τ_2) the following are equivalent:

(a) A is $ij-\delta g-$ closed.

(b) A is $ij-\delta g*-$ closed.

(c) A is $ij-g\delta s-$ closed.

(d) A is $ij-\delta gs-$ closed.

Proof: Straightforward.

Next example shows that even a $j-$ closed set in a space (X,τ_1,τ_2) need not be $ij-g\delta s-$ closed.

Example 3.5. Let $X=\{a,b,c,d,e\}$, $\tau_1\{X,\phi,\{a\},\{c\},\{b,c\}, \{a,c\},\{c,d\}, \{a,b,c\},\{b,c,d\},\{c,d,e\},\{a,c,d\},\{a,b,c,d\},\{a,c,d,e\}, \{b,c,d,e\}\}$ and $\tau_2=\{X,\phi,\{a\},\{e\},\{d\},\{a,d\},\{a,e\},\{d,e\}, \{a,c,d\},\{b,d,e\}, \{a,d,e\},\{a,b,d,e\},\{a,c,d,e\}\}$. Then $\{a,b\}$ and $\{a\}$ are $1-$ closed sets but not $21-g\delta s-$ closed sets.

Lemma 3.6. Let A be a subset of a pairwise semi regular space (X,τ_1,τ_2). Then $ji-\delta-sCl(A)=ji-sCl(A)$.

Proof: Follows from the fact that in a pairwise semi regular space, a set U is $i-$ open if and only if it is $ij-\delta-$ open.

Definition 3.7. A bitopological space (X,τ_1,τ_2) is called pairwise T_d (resp. pairwise T_b) if every $ij-gs-$closed set is $ij-g-$ closed (resp. $j-$ closed).

Theorem 3.8. Let A be a subset of a pairwise semi regular space (X,τ_1,τ_2), then

(1) A is $ij-g\delta s-$closed if and only if A is $ij-gs-$closed.

(2) If, in addition, (X,τ_1,τ_2) is $ij-T_b$ (resp. $ij-T_d$), then A is $ij-g\delta s-$ closed if and only if A is $j-$ closed (resp. $ij-g-$ closed).

Proof: Follows directly by Lemma 3.6 and Definition 3.7 above.

Theorem 3.9. For a space (X,τ_1,τ_2), the following are equivalent:

(1) Every $i-$ open set of X is $ji-\delta-$ semiclosed.

(2) Every subset of X is $ij-g\delta s-$ closed.

Proof: (1) $\Rightarrow$ (2): Let $A\subset U$, where U is $i-$ open and A is an arbitrary subset of X. By (1), U is $ji-\delta-$ semiclosed, and thus $ji-\delta-sCl(A)\subset ji-\delta-sCl(U)=U$. Hence A is $ij-g\delta s-$ closed.

(2) $\Rightarrow$ (1): If $U \subset X$ is $i-$ open, by (2), $ji-\delta sCl(U)=U$ or equivalently U is $ji-\delta-$ semiclosed.

Remark 3.10. Finite union or intersection of $ij-g\delta s-$closed sets need not be $ij-g\delta s-$closed.

Example 3.11. Let (X,τ_1,τ_2) as in Example. 3.5, then $\{a,b\}$ and $\{a,c\}$ are $12-g\delta s-$closed sets but $\{a,b\}\bigcap\{a,c\}=\{a\}$ is not $12-g\delta s-$closed.

Theorem 3.12. Let A be a subset of a bitopological space (X,τ_1,τ_2). Then we have:

(1) If A is $ij-g\delta s-$closed in X, then $ji-\delta-sCl(A)\setminus A$ does not contain any nonempty $i-$ closed set.

(2) If A is $ij-g\delta s-$closed in X and $A\subset B\subset ji-\delta-sCl(A)$, then B is $ij-g\delta s-$closed in X.

Proof: (1) Let F be an $i-$ closed set such that $F\subset ji-\delta-sCl(A)\setminus A$. Then $A\subset X\setminus F$. Since A is $ij-g\delta s-$ closed and $X\setminus F$ is $i-$ open, then $ji-\delta-sCl(A)\subset X\setminus F$ which implies $F\subset X\setminus ji-\delta-sCl(A)$. Hence $F\subset(ji-\delta-sCl(A))\bigcap(ji-\delta-sCl(A))^c=\phi$. (2) It is clear.

Corollary 3.13. If A is a $ij-g\delta s-$closed set in a space X, then A is $ji-\delta-$ semiclosed if and only if $ji-\delta sCl(A)\setminus A$ is $i-$ closed.

Theorem 3.14. If A is an $i-$ open $ij-g\delta s-$closed set in a space X, then A is $ji-$ semiclosed and thus $ij-$ regular open.

Proof. If A is $i-$ open and $ij-g\delta s-$closed, then $ji-\delta sCl(A)\subset A$ and so A is $ji-\delta-$ semiclosed. Thus A $ji-$ semiclosed, i.e. $i-Int(j-Cl(A))\subset A$. Since A is $i-$ open, then $i-Int(j-Cl(A))=A$ and A is $ij-$regular open.

Theorem 3.15. Let (X,τ_1,τ_2) be a bitopological space and $A\subset Y\subset X$. If Y is $i-$ open in X and A is $ij-g\delta s-$closed in X, then A is $ij-g\delta s-$closed relative to Y.

Proof: Let $A\subset U$, where U is $i-$ open relative to Y. Then $U=Y\bigcap G$ for some $i-$ open set G of X. Since A is $ij-g\delta s-$ closed in X and $A\subset G$, then $ji-\delta-sCl(A)\subset G$. Then

$ji-\delta-sCl_Y(A)=ji-\delta-sCl(A)\cap Y\subset G\cap Y=U$. Hence A is $ij-g\delta s-$closed relative to Y.

4- $ij-g\delta s-$ Open sets

Definition 4.1. A subset A of a bitopological space (X,τ_1,τ_2) is called $ij-g\delta-$ semi open (briefly $ij-g\delta s-$ open) if $X\setminus A$ is $ij-g\delta-$ semiclosed.

Theorem 4.2. A subset A of a bitopological space (X,τ_1,τ_2) is $ij-g\delta s-$ open if $F\subset ji-\delta-sInt(A)$ whenever F is $i-$ closed and $F\subset A$.

Proof: Follows directly from Definitions 3.1 and 4.1.

Theorem 4.3. If a subset A of a bitopological space (X,τ_1,τ_2) is $ij-g\delta s-$ open, then $U=X$ whenever U is $i-$ open and $ji-\delta-sInt(A)\cup(X\setminus A)\subset U$.

Proof: Let U be an $i-$ open set such that $ji-\delta-sInt(A)\cup(X\setminus A)\subset U$. Then $X\setminus U\subset(X\setminus ji-\delta-sInt(A))\bigcap A$, i.e., $X\setminus U\subset ji-\delta-sCl(X\setminus A))\setminus(X\setminus A)$. Since $X\setminus A$ is $ij-g\delta s-$ closed and $X\setminus U$ is $i-$ closed, then by Theorem 3.12 (1), $X\setminus U=\phi$ and hence $U=X$.

Theorem 4.4. Let (X,τ_1,τ_2) be a bitopological space, $A\subset Y\subset X$ and Y be a $ji-\delta-$ open $i-$ closed set in X. If A is $ij-g\delta s-$ open relative to Y, and then A is $ij-g\delta s-$open in X.

Proof: Let F be an $i-$ closed subset of X and $F\subset A$. Then F is $i-$ closed relative to Y and since A is $ij-g\delta s-$ open relative to Y, $F\subset ji-\delta-sInt_Y(A)=ji-\delta-sInt(A)\cap Y$. Hence $F\subset ji-\delta-sInt(A)$ and so A is $ij-g\delta s-$open in X.

Theorem 4.5. If A is an $ij-g\delta s-$open subset of a bitopological space (X,τ_1,τ_2) and $ji-\delta-sInt(A)\subset B\subset A$, then B is $ij-g\delta s-$open.

Proof: Let $F\subset B$ and F is $i-$ closed subset of X. Since A is $ij-g\delta s-$ open and $F\subset A$, we have $F\subset ji-\delta-sInt(A)$ and then $F\subset ji-\delta-sInt(B)$. Hence B is $ij-g\delta s-$open.

Theorem 4.6. If a subset A of a bitopological space (X,τ_1,τ_2) is $ij-g\delta s-$closed, then $ji-\delta-sCl(A)\setminus A$ is $ij-g\delta s-$open.

Proof: Let $F\subset ji-\delta-sCl(A)\setminus A$, where F is $i-$ closed in X. Then, by Theorem 3.12 $F=\phi$ and so $F\subset ji-\delta-sInt(ji-\delta-sCl(A)\setminus A)$. This shows that $ji-\delta-sCl(A)\setminus A$ is $ij-g\delta s-$open.

Lemma 4.7. Let A be a $ij-\delta g-$ closed subset of a bitopological space (X,τ_1,τ_2). Then $ji-\delta-Cl(A)\setminus A$ does not contain a nonempty $i-$ closed set.

Lemma 4.8. In any bitopological space (X,τ_1,τ_2) a singleton set $\{x\}$ is $ji-\delta-$ open if and only if it is $ji-$ regular open.

Definition 4.9. A bitopological space (X,τ_1,τ_2) is called pairwise $T_{\frac{3}{4}}$ if every $ij-\delta g-$ closed subset of X is $ji-\delta-$ closed.

Theorem 4.10. For a bitopological space (X,τ_1,τ_2), the following are equivalent:

(1) X is pairwise $T_{\frac{3}{4}}$.

(2) Every singleton set $\{x\}$ is either $ji-\delta-$ open or $i-$ closed.

(3) Every singleton set $\{x\}$ is either $ji-\delta-$ semi open or $i-$ closed.

(4) Every $ij-g\delta s-$closed set of X is $ji-\delta-$ semiclosed.

Proof: (1) $\Rightarrow$ (2): If $\{x\}$ is not $i-$ closed, then $X\setminus\{x\}$ is not $i-$ open and thus $ij-\delta g-$ closed. By (1), $X\setminus\{x\}$ is $ji-\delta-$ closed, i.e., $\{x\}$ is $ji-\delta-$open.

(2) $\Rightarrow$ (1): Let $A\subset X$ be $ij-\delta g-$closed. Let $x\in ji-\delta-Cl(A)$. We consider two cases:

Case (1): Let $\{x\}$ be $ji-\delta-$open. Since $x\in ji-\delta-Cl(A)$, then $\{x\}\cap A\neq\phi$. Thus $x\in A$.

Case (2): Let $\{x\}$ be $i-$ closed. If we assume that $x\notin A$, then we would have $x\in ji-\delta-Cl(A)\setminus A$ which cannot happen according to Lemma 4.7. Hence $x\in A$.

So in both cases we have $ji-\delta-Cl(A)\subset A$. Therefore A is $ji-\delta-$closed.

(2) $\Leftrightarrow$ (3): Every singleton set is $ji-\delta-$ semi open if and only if it is $ji-\delta-$ open.
(3) $\Rightarrow$ (4): Let $A\subset X$ be $ij-g\delta s-$ closed and $x\in ji-\delta-sCl(A)$. We consider the two cases:
Case (1): Let $\{x\}$ be $i-$ closed. By Theorem 3.13 $ji-\delta-sCl(A)\setminus A$ does not contain $\{x\}$. Since $x\in ji-\delta-sCl(A)$, then $x\in A$.
Case (2): Let $\{x\}$ be $ji-\delta-$ semi open. Since $x\in ji-\delta-sCl(A)$, $\{x\}\cap A\neq\phi$. This shows that $x\in A$.
So in both cases, $x\in A$. This shows that $ji-\delta-sCl(A)\subset A$. Therefore $A=ji-\delta-sCl(A)$ and A is $ji-\delta-$ semiclosed.
(4) $\Rightarrow$ (3): Let $x\in X$ and assume that $\{x\}$ is not $i-$ closed. Then clearly $X\setminus\{x\}$ is not $i-$open and $X\setminus\{x\}$ is trivially $ij-g\delta s-$ closed. By (1), it is $ji-\delta-$ semiclosed and thus $\{x\}$ is $ji-\delta-$semi open.
Definition 4.11. A subset A of a bitoplogical space (X,τ_1,τ_2) is called $ij-$ nowhere dense if $i-Int(j-Cl(A))=\phi$ and called $ij-\delta-$ nowhere dense if $i-Int(ji-\delta-Cl(A))=\phi$.
Lemma 4.12. For a bitopological space (X,τ_1,τ_2), the following are satisfied:
(a) Every singleton set $ij-\delta-$ preclosed or $ji-\delta-$ open in *X*.
(b) Every singleton set is $ij-\delta-$nowhere dense or $ij-\delta-$preopen in *X*.
Proof: (a) Let $\{x\}$ be not $ji-\delta-$ open, then $i-Cl(ji-\delta-Int(\{x\}))=\phi\subset\{x\}$ and so $\{x\}$ is $ij-\delta-$preclosed.
(b) If $\{x\}$ is not $ij-\delta-$nowhere dense, then $i-Int(ji-\delta-Cl(\{x\}))\neq\phi$. Therefore $\{x\}\subset i-Int(ji-\delta-Cl(\{x\}))$ is $ij-\delta-$preopen.
Theorem 4.13. For a bitopological space (X,τ_1,τ_2), the following are equivalent:
(a) *X* is pairwise $T_{\frac{3}{4}}$.
(b) Every $ij-\delta-$ preclosed singleton is $i-$ closed.
(c) Every non $ji-\delta-$ open singleton set of *X* is $i-$closed.

Proof: (a) $\Rightarrow$ (b): Let $x \in X$ and $\{x\}$ is $ij-\delta-$ preclosed. By Lemma 4.12 above, $\{x\}$ is not $ji-\delta-$ open and hence by Theorem 4.10, $\{x\}$ is $i-$ closed.

(b) $\Rightarrow$ (a): If $\{x\}$ is not $ji-\delta-$ open for some $x \in X$, then by Lemma 4.12, $\{x\}$ is $ij-\delta-$preclosed and by (b), $\{x\}$ is $i-$ closed. Hence X is pairwise$T_{\frac{3}{4}}$.

(b) $\Leftrightarrow$ (c): Obvious.

Recall that a bitopological space (X,τ_1,τ_2) is called pairwise $T_{\frac{1}{2}}$ [3] if every $ij-g-$ closed set is $j-$ closed.

One may notice that every pairwise T_1 space is pairwise $T_{\frac{3}{4}}$ and every pairwise $T_{\frac{3}{4}}$ space is pairwise $T_{\frac{1}{2}}$ but not conversely.

Lemma 4.14. For a bitopological space(X,τ_1,τ_2), the following are equivalent:

(a) Every $ji-\delta-$ preopen singleton set is $i-$ closed.

(b) Every singleton set is $ij-\delta-$ nowhere dense or $i-$ closed.

Proof: (a) $\Rightarrow$ (b): By Lemma 4.12, every singleton is either $ij-\delta-$ nowhere dense or $ji-\delta-$ preopen. In the first case we are done, in the second case $i-$ closeness follows from assumption.

(b) $\Rightarrow$ (a): Let $\{x\}$ be $ji-\delta-$ preopen. Assume that $\{x\}$ is not $i-$ closed. Then by (b) it is $ij-\delta-$ nowhere dense. Thus $\{x\} \subset i-Int(ji-\delta-Cl(\{x\})=\phi$, which is impossible.

Theorem 4.15. For a bitopological (X,τ_1,τ_2), the following are equivalent:

(a) X is pairwise $T_{\frac{3}{4}}$.

(b) X is pairwise and every $j-$ open singleton set is $ji-\delta-$ open.

Proof: Obvious.

Theorem 4.16. Let A be an $ij-\delta gs-$ closed subset of a bitopological (X,τ_1,τ_2). Then $ji-\delta-sCl(A)\setminus A$ does not contain a nonempty $ij-\delta-$ closed set.

Proof: Similar to that of Theorem 3.12.

Definition 4.17. A bitopological space (X,τ_1,τ_2) is called pairwise almost weakly Hausdorff if (X,τ_1^*,τ_2^*) is pairwise $T_{\frac{1}{2}}$.

Theorem 4.18. For a bitopological (X,τ_1,τ_2), the following are equivalent:
(a) X is pairwise almost weakly Hausdorff.
(b) Every singleton set of X is $ij-\delta-$ closed or $ji-\delta-$ semiopen.
(c) Every $ij-\delta gs-$closed set of X is $ji-\delta-$ semiclosed.

Proof: (a) $\Leftrightarrow$ (b): Follows from the fact that every singleton set is $ij-\delta-$ semiopen if and only if it is $ij-\delta-$ open.

(b) $\Rightarrow$ (c): Let $A\subset X$ be an $ij-\delta gs-$ closed and $x\in ji-\delta-sCl(A)$. We consider the following two cases:
Case(1): Let $\{x\}$ be $ji-\delta-$ semiopen. Since $x\in ji-\delta-sCl(A)$, then $\{x\}\bigcap A\neq\phi$. This shows that $x\in A$.
Case(2): Let $\{x\}$ be $ij-\delta-$ closed. If we assume that $x\notin A$, then we have $x\in ji-\delta-sCl(A)\setminus A$ which cannot happen according to Theorem 4.17. Hence $x\in A$.
So in both cases we have $ji-\delta-sCl(A)\subset A$, and so A is $ji-\delta-$ semiclosed.

(c) $\Rightarrow$ (b): If $\{x\}$ is not $ij-\delta-$ closed, then $X\setminus\{x\}$ is not $ij-\delta-$ open and thus $X\setminus\{x\}$ is $ij-\delta gs-$ closed. By (c), $X\setminus\{x\}$ is $ji-\delta-$ semiclosed, i.e., $\{x\}$ is $ji-\delta-$ semiopem.

5- $ij-g\delta s-$ Continuous and $ij-g\delta s-$irresolute functions

Definition 5.1. A function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is called:
(1) $ij-$ super continuous [9] (resp. $ij-\delta-$ semi continuous, $ij-\delta-$ semi irresolute) if $f^{-1}(F)$ is an $ij-\delta-$ closed (resp. $ij-\delta-$ semiclosed, $ij-\delta-$ semiclosed) set in X for every $i-$ closed (resp. $i-$ closed, $ij-\delta-$ semiclosed) set F in Y.
(2) $ij-g-$ continuous [3] (resp. $ij-gs-$ continuous [8]) if $f^{-1}(F)$ is $ij-g-$ closed (resp. $ij-gs-$ closed) in X for every $j-$ closed set F of Y.

(3) $ij-\delta g-$ Continuous (resp. $ij-\delta g-$ irresolute) if $f^{-1}(F)$ is $ij-\delta g-$ closed in X for every $j-$ closed (resp. $ij-\delta g-$ closed) set F of Y.

(4) $ij-g\delta-$ continuous (resp. $ij-g\delta-$ irresolute) if $f^{-1}(F)$ is $ij-g\delta-$ closed in X for every $j-$ closed (resp. $ij-g\delta-$ closed) set F of Y.

(5) $ij-\delta gs-$ continuous if $f^{-1}(F)$ is $ij-\delta gs-$closed set in X for every $j-$ closed set F of Y.

(6) $ij-\delta-$ semiclosed (resp. $ij-\delta-$ semiopen) if $f(F)$ is $ij-\delta-$ semiclosed (resp. $ij-\delta-$ semiopen) in Y for every $ij-\delta-$ semiclosed (resp. $ij-\delta-$ semiopen) set F of X.

Definition 5.2. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $ij-g\delta s-$ continuous if $f^{-1}(V)$ is $ij-g\delta s-$closed in X for every $j-$ closed set V of Y. If f is $12-g\delta s-$continuous and $21-g\delta s-$continuous, then it is called pairwise $g\delta s-$continuous.

Definition 5.3. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $ij-g\delta s-$ irresolute if $f^{-1}(V)$ is $ij-g\delta s-$closed in X for every $ij-g\delta s-$ closed set V of Y. If f is $12-g\delta s-$irresolute and $21-g\delta s-$ irresolute, then it is called pairwise $g\delta s-$ irresolute.

Clearly $f:X\to Y$ is $ij-g\delta s-$continuous (resp. $ij-g\delta s-$ irresolute) if and only if $f^{-1}(V)$ is $ij-g\delta s-$open in X for every $j-$ open (resp. $ij-g\delta s-$ open) set V of Y.

Example 5.4. Let (X,τ_1,τ_2) as in Example 3.5. Then,

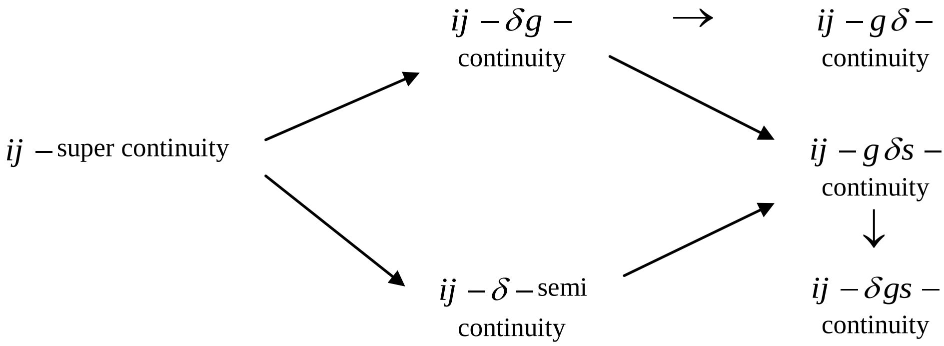
(a) The function $f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $f(a)=b$, $f(b)=a$, $f(c)=c$, $f(d)=f(e)=e$ is not $12-g\delta s-$ continuous, since $\{e\}$ is a $2-$ closed set but $f^{-1}(\{e\})=\{d,e\}$ is not $12-g\delta s-$ closed.

(b)The function $g:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $g(a)=g(b)=g(c)=g(e)=e$, $g(d)=b$ is $12-g\delta s-$ continuous.

(c) The function $h:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $h(a)=h(c)=c$, $h(b)=h(d)=b$, $h(e)=e$ is $12-g\delta s-$irresolute.

A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called pairwise continuous [4] if the induced functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are continuous.

Remark 5.5. (a) For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, we have the following diagram:



(b) None of these implications is reversible.

(c) The notions of $ij-g\delta-$continuity and $ij-g\delta s-$continuity are independent of each other.

(d) The notions of $ij-g\delta-$irresoluteness, $ij-\delta g-$ irresoluteness and $ij-g\delta s-$irresoluteness are mutually independent.

The following examples show that the inverses of the implication in the above diagram may not be satisfied.

Example 5.6. Let (X,τ_1,τ_2) be as in Example 3.5, then

1- the function $f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $f(a)=f(c)=f(d)=b$, $f(b)=a$, $f(e)=e$ is not $12-g\delta-$ continuous, since $\{b\}$ is $2-$ closed but $f^{-1}(\{b\})=\{a,c,d\}$ not $12-g\delta-$ closed.

2- the function $f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $f(a)=c$, $f(b)=b$, $f(c)=a$, $f(d)=d$,$f(e)=e$ is $12-g\delta-$ continuous.

3- the identity function $f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ is $12-\delta g-$ irresolute.

4- the function $f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ defined as $f(a)=b$, $f(b)=d$, $f(c)=a$, $f(d)=c$, $f(e)=e$ is not $12-\delta g-$ irresolute, since $\{a,c\}$ is $12-\delta g-$ closed set but $f^{-1}(\{a,c\})=\{c,d\}$ is not $12-\delta g-$closed.

Theorem 5.7. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function, then

(a) If *f* is $ij-g\delta s-$irresolute and *X* is pairwise $T_{\frac{3}{4}}$, then *f* is $ji-\delta-$ semi irresolute.

(b) If *f* is $ij-g\delta s-$continuous and *X* is pairwise $T_{\frac{3}{4}}$, then *f* is $ij-\delta-$ semi continuous.

(c) If *X* is pairwise semi regular, then *f* is $ij-g\delta s-$continuous if and only if *f* is $ij-gs-$ continuous.

(d) If *X* is pairwise semi regular and pairwise T_b (resp. pairwise T_d), the *f* is pairwise $g\delta s-$continuous if and only if *f* is pairwise continuous (resp. pairwise $g-$ continuous).

Proof: (a) Let *V* be a $ji-\delta-$ semiclosed set in *Y*. Then *V* is $ij-g\delta s-$ closed in *Y*, and since *f* is $ij-g\delta s-$irresolute, then $f^{-1}(V)$ is $ij-g\delta s-$ closed in *X*. Since *X* is pairwise $T_{\frac{3}{4}}$, $f^{-1}(V)$ is $ji-\delta-$ semiclosed in *X*. Hence *f* is $ji-\delta-$semi irresolute.

(b) Similar to (a).

(c) and (d) follows from Theorem 3.8.

Theorem 5.8. If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a pairwise continuous $ji-\delta-$ semiclosed function, then $f(A)$ is $ij-g\delta s-$closed in *Y* for every $ij-g\delta s-$closed set *A* in *X*.

Proof: Let *A* be a $ij-g\delta s-$closed set in *X*. Let $f(A)\subset V$, where *V* is any $i-$ open set in *Y*. Since *f* is pairwise continuous, $f^{-1}(V)$ is $i-$ open in *X* and $A\subset f^{-1}(V)$. Then we have $ji-\delta-sCl(A)\subset f^{-1}(V)$ and so $f(ji-\delta-sCl(A))\subset V$. Since *f* is $ji-\delta-$ semiclosed, $f(ji-\delta-sCl(A))$ is $ji-\delta-$semiclosed in *Y* and hence $ji-\delta-sCl(f(A))\subset ji-\delta-sCl(f(ji-sCl(A))\subset V$. This shows that $f(A)$ is $ij-g\delta s-$closed in *Y*.

Theorem 5.9. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ and $g:(Y,\sigma_1,\sigma_2)\to(Z,\nu_1,\nu_2)$ be two functions, where (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,ν_1,ν_2) are bitopological spaces. Then,

(a) If *f* is $ij-g\delta s-$ continuous and *g* is pairwise continuous, then $g\circ f$ is $ij-g\delta s-$continuous.

(b)) If *f* is irresolute and *g* is $ij-g\delta s-$ irresolute, then $g\circ f$ is $ij-g\delta s-$ $ij-g\delta s-$irresolute.

(c) If *f* is $ij-g\delta s-$ irresolute and *g* is $ij-g\delta s-$ continuous, then $g\circ f$ is $ij-g\delta s-$continuous.

(d) Let (X,τ_1,τ_2) be a pairwise $T_{\frac{3}{4}}$ space. If *f* is $ji-\delta-$ semi irresolute and *g* is $ij-g\delta s-$continuous, then $g\circ f$ is $ji-\delta-$ semi continuous.

Proof: Obvious.

Theorem 5.10. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise continuous $ji-\delta-$ semiclosed surjection. If (X,τ_1,τ_2) is a pairwise $T_{\frac{3}{4}}$ space, then (Y,σ_1,σ_2) is pairwise$T_{\frac{3}{4}}$.

Proof: Let *F* be a $ij-g\delta s-$closed set of *Y*. Then, by Theorem 5.8 $f^{-1}(F)$ is $ij-g\delta s-$closed in *X*. Since *X* is pairwise$T_{\frac{3}{4}}$, then $f^{-1}(F)$ is $ji-\delta-$ semiclosed in *X*. By the rest of the assumption it follows that *F* is $ji-\delta-$ semiclosed in *Y*. Hence *Y* is pairwise$T_{\frac{3}{4}}$.

Theorem 5.11. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $j-$ closed (i.e., the image of each $j-$closed set of *X* is $j-$ closed in *Y*) and $ij-\delta-$ semi open bijection. If X is a pairwise $T_{\frac{1}{2}}$ space, then *Y* is pairwise $T_{\frac{3}{4}}$.

Proof: Let$y\in Y$. Since *X* is pairwise $T_{\frac{1}{2}}$ and *f* is bijective, then for some $x\in X$ with$f(x)=y$, we have $\{x\}$ is $j-$ closed or $i-$ open. If $\{x\}$is $j-$ closed then $\{y\}=f(\{x\})$ is $j-$closed, since *f* is $j-$closed and injective. If $\{x\}$ is $i-$open, then $\{y\}$ is $ij-\delta-$ semiopen, since *f* is $ij-\delta-$semiopen. Hence *Y* is pairwise$T_{\frac{3}{4}}$.

In the end of this section we define the concept of pairwise $g\delta sc-$homeomorphism and prove that the set of all pairwise $g\delta sc-$homeomorphisms from (X,τ_1,τ_2) onto itself has a group structure under the composition.

Definition 5.12. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called pairwise $g\delta sc-$homeomorphism if f is a bijective pairwise $g\delta s-$irresolute and its inverse function f^{-1} is pairwise $g\delta s-$irresolute.

For a bitopological space (X,τ_1,τ_2), we introduce the following notations:

$$g\delta sch(X,\tau_1,\tau_2)=\{f : f:(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2) \text{ is a pairwise } g\delta sc-\text{homeomorphism}\}$$

Theorem 5.13. (a) The set $g\delta sch(X,\tau_1,\tau_2)$ is a group which contains $h(X,\tau_1,\tau_2)$ as its subgroup.

(b) If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a pairwise $g\delta sc-$homeomorphism, then f induces an isomorphism from the group $g\delta sch(X,\tau_1,\tau_2)$ onto $g\delta sch(Y,\sigma_1,\sigma_2)$.

Proof: (a) A binary operation $\mu: g\delta sch(X,\tau_1,\tau_2)\times g\delta sch(X,\tau_1,\tau_2)\to g\delta sch(X,\tau_1,\tau_2)$ is well defined by $\mu(a,b)=b\circ a$ (the composition) for any $a,b\in g\delta sch(X,\tau_1,\tau_2)$. Then, it is shown that $g\delta sch(X,\tau_1,\tau_2)$ is group with binary operation μ. Every homeomorphism is both pairwise continuous and $ij-\delta-$ semiclosed. By Theorem 5.8, every pairwise homeomorphism is pairwise $g\delta sc-$homeomorphism. Therefore it is shown that $h(X,\tau_1,\tau_2)$ is a subgroup of $g\delta sch(X,\tau_1,\tau_2)$.

(b)The isomorphism $f_*: g\delta sch(X,\tau_1,\tau_2)\to g\delta sch(Y,\sigma_1,\sigma_2)$ is induced from f by $f_*=f\circ\rho\circ f^{-1}$ for every $\rho\in g\delta sch(X,\tau_1,\tau_2)$ by usual argument (the argument obtained in group theory).

REFERENCES

[1] Banerjee, G.K. On pairwise almost strongly $\theta-$ continuous mappings, Bull. Calcutta Math. Soc. 79(1987) 314-320.

[2] Bose, S., Semi open sets, semicontinuity and semi open mappings in bitopological spaces, Bull. Calcutta Math. Soc., 73, (1981) 237-246.

[3] Fukutake, T., On generalized closed sets in bitopological spaces, Bull. Fukuoko Univ. Ed. III 35 (1986) 19-28.

[4] Kelly, J.C.. Bitopological spaces, Proc. London Math. Soc., 13 (1963) 71-89.

[5] Khedr, F. H., $c\alpha-$ continuity in Bitopological spaces, Arabian J, for Science and Engineering, Volume 17, Number 1 (January 1992) 85-89.

[6] Khedr, F. H., Properties of $ij-\delta-$open sets, Fasciculi Mathematici , Nr 52 (2014), 65 - 81.

[7] Khedr, F.H. and Al-Areefi, S.M. , $\theta-$connectedness and $\delta-$ connectedness in bitopological spaces, The Arabian J. for Sci. and Engineering, Volume 18 Number 1 (1993) 51-56.

[8] Khedr, F.H. and H.S.Alsaadi, Generalized semi-closed functions in bitopological spaces, J. Egyptian Math. Soc. 20 (2012), 14-19.

[9] Khedr, F.H.; Al-Shibani, A.M., On pairwise suber continuous mappings in bitopological spaces, Internat. J. Math. & Math. Sci. Vol. 14 No. 4 (1991), 715-722.

[10] Khedr, F.H.; Al-Shibani, A.M. and Noiri, T., On δ-continuity in bitopological spaces, J. Egyptian Math. Soc. Vol.5 (1) 57-63(1997).

[11] Reilly,I.L., On Bitopological separation Properties, Nanta Math., 2(1972) 14-25.

[12] Singal, A.S. and Arya, A.S., On pairwise almost regular spaces, Glasnik Mat. 6(26)(1971), pp. 335-343.

المجموعات المعممة نصف المغلقة من النوع دلتا
في الفضاءات ثنائية التوبولوجي

***فتحي هشام خضر**

**قسم الرياضيات- كلية العلوم- جامعة اسيوط*

الهدف من هذا البحث هو تقديم ودراسة المجموعات المعممة نصف المغلقة من النوع دلتا ودراسة خواصها وعلاقاتها بالأنواع الأخرى من المجموعات المغلقة المعممة في الفضاءات ثنائية التوبولوجي. باستخدام هذه المجموعات نعطي تشخيصا للفضاء ثناثي التوبولوجي $T_{\frac{3}{4}}$. أيضا نقدم مفهوم الدوال بين الفضاءات ثنائية التوبولوجي المتصلة من النوع $g\delta s$ والمترددة من النوع $g\delta s$. أخيرا نقدم مفهوم التشاكل بين الفضاءات التوبولوجية من النوع $g\delta sc$ ونبين أن مجموعة كل التشاكلات من النوع $g\delta sc$ من فضاء ثنائي التوبولوجي إلى نفسه تكون زمرة مع عملية تحصيل الدوال.