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# THE LIGHT CONE OF THE ANTI MACH SPACE-TIME

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**The metric of the anti Mach light cone is presented, the rotation coefficients are calculated and its behavior at the different singular points on the light cone are studied, the local differential invariants of the second order are determined and the relation between sky coordinates and the coordinates of the cone parametrs are derived.**

## 1- INTRODUCTION

The most important features of any space-time is the existence of null curves, particularly null geodesic, null surfaces and null hypersurfaces which are characteristics for the Einstein field equations. The existence of these lightlike manifolds have physical origin, since null geodesic are trajectories of the photons. The null hypersurfaces in geometric optics can be considered as level surfaces (null surfaces). According to the general theory of relativity, the null geodesic constituting the light cone with vertex $P$ in space time manifold $M$ can be forced to reconverge by sufficiently strong gravitational field, e.g. quasar, galaxy or closter of galaxies. The phenomenon is known as lense effect, which is today one of the most growing areas in astrophysics ([1]- [4]). This paper is organized as follow: The sturcture of the light cone equations and its singularities are given in § 2. The differential invariants and rotation coefficients are studied in § 3. In § 4 we gives the relation between sky coordinates and the coordinates of the cone parametrization.

## 2- LIGHT CONE EQUATIONS

During the investigation of the solutions of the Einstein's field equation which allow a transitive group of motions. Ozsváth and Schu č king [5] found a vacuum solution of the field equations .

The line element for the anti-Mach metric is given by:

$$ds^2 = dx^2 - 4t\,dx\,dz + 2dy\,dz + 2t^2 dz^2 + dt^2.$$

This curious solution contradicts the known "Mach principle 3" which state that: "In the absence of matter, space-time should necessarily be Minkowsky" [6], since if we interpret the absence of matter as the absence of singularities in the solution of Einstein's vacuum field equations for gravitational fields, then this "Mach-3" principle is, as we see from the above metric, not valid.

The curvature tensor for the metric can be calculated and has the form

$$R_{ijkl} = 4(\delta^3_{[i}\delta^4_{j]}\delta^3_{[k}\delta^4_{l]} - \delta^3_{[i}\delta^1_{j]}\delta^3_{[k}\delta^1_{l]}), \quad (2.2)$$

where the brackets around the indices indicate the skew symmetric part of an expression. Its easy to see that

$$g^{ik}R_{ijkl} = R_{jl} = 0, \quad (2.3)$$

which emphasize that (2.1) is a solution of the Einstein's field equations with the curvature tensor different from zero.

The structure of the null geodesics (light rays) emitted from the vertex of the light cone are given by:

$$x = -2\alpha w + \sqrt{2}\beta(1-\cos 2w) + 2\alpha \sin 2w, (2.4)$$

$$y = \frac{-1}{4\sqrt{2}} \sin 4w\, (\sqrt{2}\,\beta - 2\alpha \tan w)^2 + \sqrt{2}\,\alpha^2 (w - 2\tan w),$$

$$z = \sqrt{2}\, w, \quad (2.6)$$

$$t = -\sqrt{2}\,\alpha(1-\cos 2w) + \beta \sin 2w, \quad (2.7)$$

where $w$ is an affine parameter along the null geodesic, $\alpha$ and $\beta$ are directional parameters which determine a unique null geodesic on the light cone and have the range $(-\infty < \alpha, \beta < \infty)$.

The light cone given by the equations (2.4) - (2.7) become singular where the matrix of the derivatives

$$\left\| \frac{\partial x^\mu}{\partial \alpha}, \frac{\partial x^\mu}{\partial \beta}, \frac{\partial x^\mu}{\partial w} \right\|$$

has a rank $< 3$, that is

$$\frac{\partial z}{\partial w}\left(\frac{\partial x}{\partial \alpha}\frac{\partial t}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial t}{\partial \alpha}\right)$$

$$= 2\sqrt{2}[(\sin 2w - w)\sin 2w + (1-\cos 2w)^2] = 0.$$

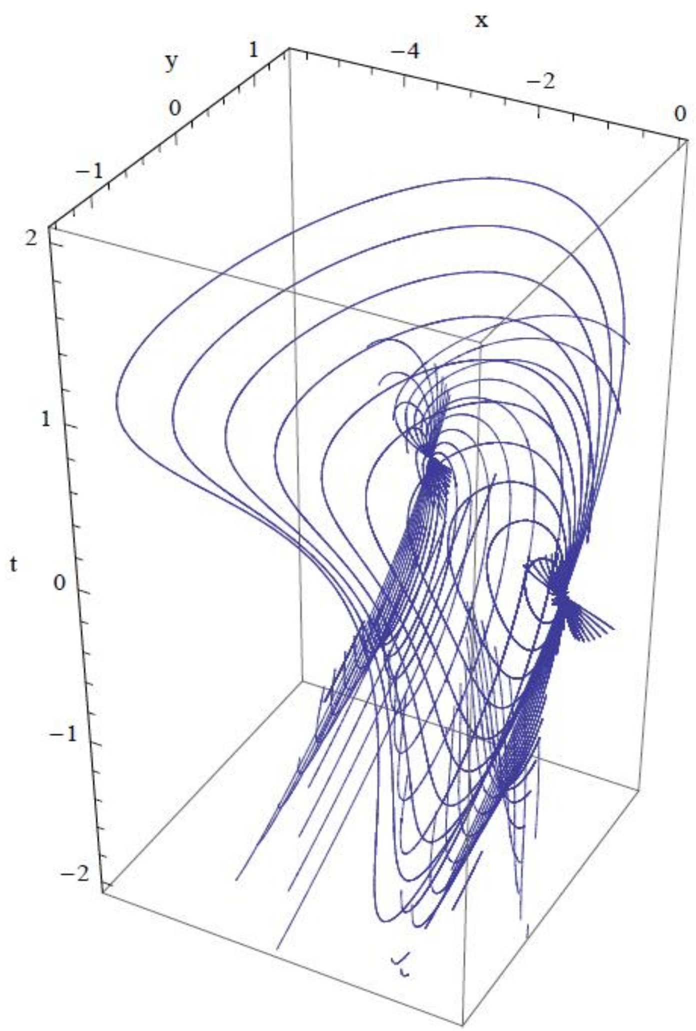


**Figure 1**: Null geodesics emitted from the vertex with the z- coordinate is suppressed. The affine parameter $w$ ranges from 0 (vertex) to $2\pi$ , the directional parameter $\alpha$ taken from 0 to 2 with constant separation 0.5 and the directional parameter $\beta$ ranged from -2 to 2 with constant separation 0.2. Keels line appear as parabols in the $(x, y)$ plane.

Two an infinite number of $w$ - values indicated from equation (2.8), namely

$$w = k\,\pi,\ k = 0,1,2,... \qquad (2.9)$$

and

$$\tan w = \frac{w}{2}. \qquad (2.10)$$

The values of $w$ given by the equation (2.9) called *points of the first kind*, and the positive solutions of the equation (2.10) are *points of the second kind*. The values of $w$ satisfies the equation (2.10) can be given by using the Mathematica program as the intersection points of the two curves $\tan w$ and $\frac{w}{2}$ and read as:

$$w = 0,\ 4.27478,\ 7.59655, 10.8127, 13.9952, 17.1628,\ 20.3223,...$$

The structures of the light cone at the points given by (2.9) are read as:

$$(x, y, z, t) = (-2\pi\, k\,\alpha,\ \sqrt{2}\,\alpha^2 k\pi, \sqrt{2}\pi\, k, 0), (2.12)$$

that is

$$x^2 = 2\sqrt{2}\,\pi\, k\, y, \qquad z = \sqrt{2}\,\pi\, k \quad \text{(Constant)}$$

The singularities at $w = k\pi$ are called *Keel points* and the lines segments of these point are called Keel lines. From equation (2.13) the Keel lines of the Anti Mach light cone are set of the parabolas in the $(x,y)$ plane (fig.1.). Keel lines are space like curves with parameter $\alpha$ which are not geodesics, to see this. The components of tangential vector $\frac{dx^\mu}{d\alpha}$ are

$$\frac{dx^\mu}{d\alpha} = (-2\pi k,\ 2\sqrt{2}\alpha\pi k, 0, 0), \qquad (2.14)$$

The normal to the Keel is the timelike unit vector

$$\frac{1}{\sqrt{1+2\alpha^2}}[\sqrt{2}\,\alpha, 1, 0, 0]. \qquad (2.15)$$

The curvature is given by

$$\kappa = \frac{1}{\sqrt{2}\pi\, k\,(1+2\,\alpha^2)}\ . \qquad (2.16)$$

At the points of the second kind, we get:

$$x = 2\sin^2 w\,(\sqrt{2}\,\beta - \alpha\, w) \qquad (2.17)$$

$$y = \frac{-1}{4\sqrt{2}}\sin 4w\,(\sqrt{2}\,\beta - \alpha\, w)^2 \qquad (2.18)$$

$$z = \sqrt{2}\, w \qquad (2.19)$$

$$t = \frac{1}{\sqrt{2}}\sin 2w\,(\sqrt{2}\beta - \alpha w). \qquad (2.20)$$

For every positive solution given by the equation (2.11) we get a surface with two directional parameters $\alpha$ and $\beta$.

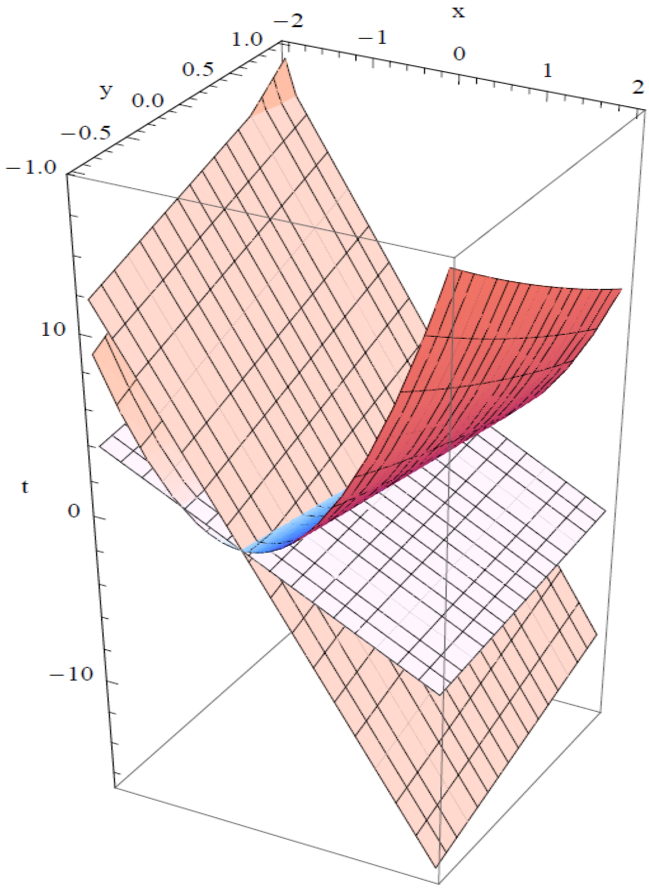


**Figure 3:** Null surface according to the parametrization given by the equations (2.17)- (2.20) at $w = 4.27478227\,1458128$. Plotted are the coordinates $x(\alpha,\beta)$, $y(\alpha,\beta)$ and $t(\alpha,\beta)$, z coordinates is suppressed. The two directional parameters $\alpha$ and $\beta$ are arranged from -2 to 2.

## 2.1 THE INNER METRIC

The intrinsic three- dimensional metric of the light cone can be formed from the relation

$$\gamma_{lk} = g_{ij}\frac{\partial x^i}{\partial y^l}\frac{\partial x^j}{\partial y^k}, \qquad (2.21)$$

where $y^l = (w, \alpha, \beta)$. Since $\frac{\partial x^i}{\partial w}$ is the tangential vector to the cone, the component of the cone metric reduced to a two dimensional metric:

$$\gamma_{1A} = 0, \quad A = 1,2,3 \tag{2.22}$$

$$\gamma_{22} = 4(\sin 2w - w)^2 + 2(1 - \cos 2w)^2 \tag{2.23}$$

$$\gamma_{23} = \sqrt{2}\,(1 - \cos 2w)(\sin 2w - 2w) \tag{2.24}$$

$$\gamma_{33} = 2(1 - \cos 2w)^2 + \sin^2 w. \tag{2.25}$$

The determinant of the inner metric $\Delta$ is given by:

$$\Delta = \gamma_{22}\ \gamma_{33} - (\gamma_{23})^2 = 16(w\cos w - 2\sin w)^2 \sin^2 w.$$

The set of singular points given by the two equations (2.9) and (2.10) can be also determined by putting $\Delta = 0$.

## 3- ROTATION COEFFICIENTS AND DIFFERENTIAL INVARIANTS

The local differential geometry of null hypersurfaces are studied in some detail described in [ 7], [8], [9], [10], [11] and [12] . In the triad formalism the light cone metric can be represented as:

$$\gamma_{i\,k} = t_i\,\bar{t}_k + \bar{t}_i\,t_k\,, \tag{3.1}$$

where $t_i$ is a complex covariant vector intrinsic to the cone.

The rotation coefficients : divergence $\rho$, shear $\sigma$ and torsion $\tau$ are given in terms of the triad and its derivatives by:

$$\rho + i\upsilon = \varepsilon^i t^k (\bar{t}_{i,k} - \bar{t}_{k,i}). \tag{3.2}$$

$$\sigma = \varepsilon^i \bar{t}^k (\bar{t}_{i,k} - \bar{t}_{k,i}). \tag{3.3}$$

$$\tau = \bar{t}^i t^k (\bar{t}_{i,k} - \bar{t}_{k,i}). \tag{3.4}$$

where $\varepsilon^i$ is the generator of light cone and satisfies $\gamma_{ik}\,\varepsilon^k = 0$ and can be chosen as.

$$\varepsilon^i = \delta^i_1.$$

Comparing equation (3.1) with equations (2.22) - (2.25) we get:

$$t_1 = 0 \tag{3.5}$$

$$t_2 = \sqrt{2}\,(\sin 2w - w) + i\,(1 - \cos 2w) \tag{3.6}$$

$$t_3 = (1 - \cos 2w) - \frac{i}{\sqrt{2}} \sin 2w, \tag{3.7}$$

The contravariant component $t^i$ are calculated from $t_i = \gamma_{ik}\,t^k$, then:

$$t^1 = 0 \tag{3.8}$$

$$t^2 = \frac{2i+\sqrt{2}\,cot\ w}{8-4\ w\ cot\ w}, \tag{3.9}$$

$$t^3 = \frac{csc^2\ w\ (-1-i\sqrt{2}\ \ w+cos\,2\,w+i\sqrt{2}\ \ sin\,2\,w)}{-8+4\,w\ cot\ w}. \tag{3.10}$$

The non vanishing rotation coefficient can be found from:

$$\rho + i\upsilon = -t^2\bar{t}_{2,1} - t^3\bar{t}_{3,1}. \tag{3.11}$$

$$\sigma = -\bar{t}^{\,2}\bar{t}_{2,1} - \bar{t}^{\,3}\bar{t}_{3,1}. \tag{3.12}$$

From equations (3.11), (3.12) and (3.5) - (3.10) we get:

$$\rho = \frac{w+cot\ w\ (3-w\,cot\ w)}{2w\ cot\ \ w-4} \tag{3.13}$$

$$\upsilon = \sqrt{2} \tag{3.14}$$

$$\sigma = \frac{csc^2\ w}{4\,w\ cot\ w-8}\ (2i\sqrt{2}+2(-i\sqrt{2}+w)\ cos\,2w-(1+2\sqrt{2}i\,w)\ sin\,2w) \tag{3.15}$$

$$|\sigma|^2 = \sigma\bar{\sigma} = \frac{1}{4(-2+w\ cot\ \ w)^2}(7-4\,w^2+csc^2\ w+w\ (w\ csc^2\ w$$

$$(4+csc^2\ \ w)-2\ cot\ \ w\ (6+csc^2\ w))). \tag{3.16}$$

The behavior of $\rho, \sigma$ and $|\sigma|$ at the vertex of the light cone and at different singular points can introduced by using Mathematica program with help of the two equations (2.9) and (2.11) (Also can be indicated from the figures below) as follows: $\rho$ tends to $\pm\infty$ at the vertex $w = 0$ and at $w = k\pi$ ,

$$w = \tan\frac{w}{2}\,.$$

$\sigma$ and $|\sigma|^2$ tend to zero at the vertex and tend to infinity at the points of first and second kind.

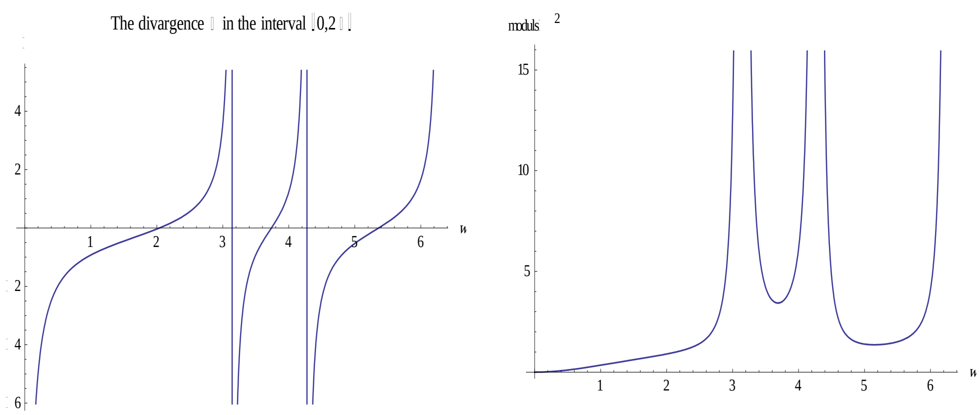


**Figure 3**

The values of $\rho$ and $|\sigma|^2$ evaluated in the two equations (3.13) and (3.16) can be checked by using the two equations

$$\rho = -\frac{1}{4\Delta}\frac{d\Delta}{dw}, \quad |\sigma|^2 = \rho^2 - \frac{\det\left(\frac{d\gamma_{ik}}{dw}\right)}{4\Delta}. \qquad (3.17)$$

There exists in general one first order invariant of a null hypersurface, i.e. an invariant function formed from the rotation coefficient alone, without derivative. This is the quantity $j = \frac{\rho}{|\sigma|}$. The complex invariant $j$ has the dimension (length)$^{-1}$, is nonlinear in the second order derivatives of the inner metric and involves additionally transversal derivatives. $j$ describes changes of the nullsurface geometry in transversal directions [8].

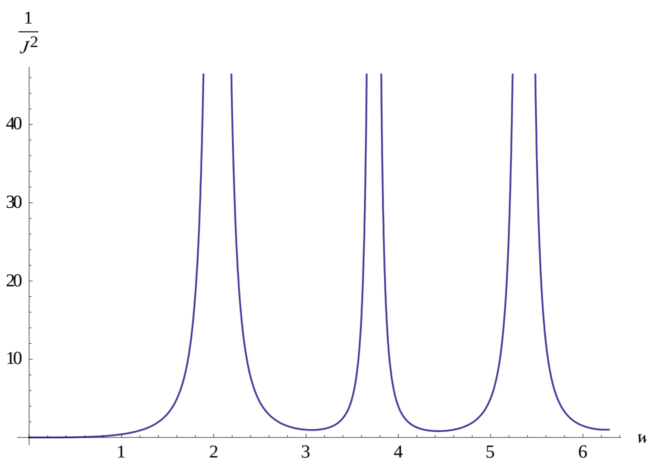


**Figure 4**

It is important to consider $\frac{1}{j^2}$ which measure the anisotropic behavior of the generators around a given one:

$$\frac{1}{j^2} = \frac{7 - 4w^2 + (1+4w^2)\csc^2 w + w^2\csc^4 w - 2w\cot w\,(6+\csc^2 w)}{(w + 3\cot w - w\cot^2 w)^2} \qquad (3.18)$$

At the vertex ($w = 0$) $\frac{1}{j^2}$ tend to zero, and at the points of the first and second kind $\frac{1}{j^2}$ tends to $1$.

Another invariant can be obtained in the inner geometry by taking a certain linear combination of these affine invariants [8]. If we define

$$I = I_1 + iI_2 = i\left(\frac{\omega}{\rho|\sigma|} - \frac{\psi}{\sigma|\sigma|} + \frac{1}{j} - j\right) = \frac{i}{|\sigma|}\left(\frac{D\rho}{\rho} - \frac{D\sigma}{\sigma}\right) + \frac{2\upsilon}{|\sigma|}, \qquad (3.19)$$

short calculations give:

$$|\sigma|^3\ I_1 = \frac{2\sqrt{2}\ (w\cos w - \sin w)}{w\cos w - 2\sin w}, \qquad (3.20)$$

$$|\sigma|^3\ I_2 = \frac{\csc^4 w\,(-44 - 48w^2 + M_1 + M_2)}{16\,(-2 + w\cot w)^2\,(2\,w\cos 2w - 3\sin 2w)}, \qquad (3.21)$$

where

$$M_1 = (63 + 48w^2)\cos 2w + 4(-5 + 4w^2)\cos 4w + \cos 6w,$$

$$M_2 = 48w\sin 2w + 32w^3\sin 2w - 32w\sin 4w.$$

$I_1$ is a measure for the rotation of the shear directions (i.e. directions where the distance change to neighbouring rays is a maximum or minimum) relative to the generator congruence . $I_2 = D\,j/\rho$ describes the change of the first-order quantity $j$ along the rays. The behavior at the vertex and at the different singularity points for the invariant $I_1$ and $I_2$ can be given from mathematica program or from the figure 5 as follows: $I_1$ and $I_2$ tend to $\pm$ infinity at the vertex and to zero at the other points of singularities.

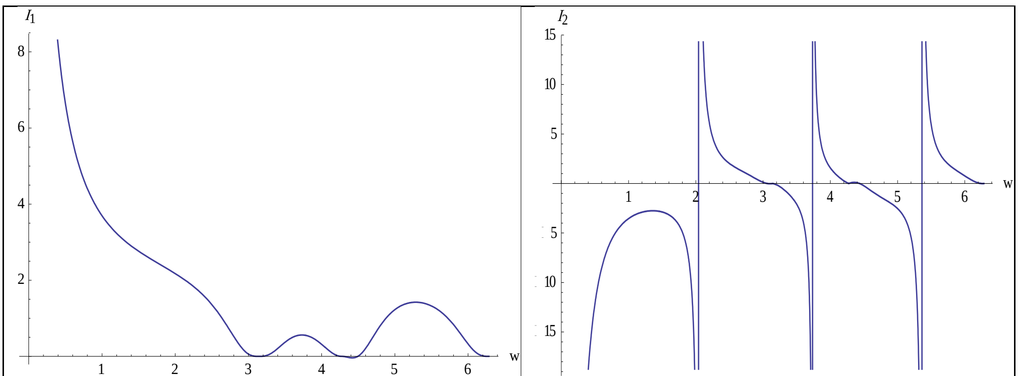


**Figure 5**

## 4- OBSERVER SKY AND CONE PARAMETRIZATION

In sky coordinate $\theta$ and $\phi$ any cone metric can be expanded in powers of an affine parameter $w^*$ near the vertex as [2].

$$\gamma^*_{\theta\theta} = \frac{(w^*)^2}{2} + O(w^*)^4 \qquad (4.1)$$

$$\gamma^*_{\theta\phi} = O\ (w^*)^5 \qquad (4.2)$$

$$\gamma^*_{\phi\phi} = \frac{(w^*)^2}{2}\sin^2\theta + O(w^*)^4 \qquad (4.3)$$

In an anti Mach light cone from equations (2.23) - (2.25) we have:

$$\gamma_{22} = 4w^2 + O(w^4) \tag{4.4}$$

$$\gamma_{23} = O(w^5) \tag{4.5}$$

$$\gamma_{33} = 4w^2 + O(w)^4. \tag{4.6}$$

The coordinate $(\alpha, \beta, w)$ are related to $(\theta, \phi, w^*)$ and this coordinate transformation should take approximately the form $\theta = \theta(\alpha, \beta)$ , $\phi = \phi(\alpha, \beta)$ and $w^* = \dfrac{w}{m(\alpha, \beta)}$ near the vertex. i.e for small $w$ we have

$$\gamma_{ij} = \gamma^*_{\mu\nu} \frac{\partial x^\mu}{\partial x^i} \frac{\partial x^\nu}{\partial x^j}. \tag{4.7}$$

From equations (4.1)- (4.3) and (4.4) - (4.6) with equation (4.7) we get:

$$(\frac{\partial \theta}{\partial \alpha})^2 + (\frac{\partial \phi}{\partial \alpha})^2 \sin^2\theta = 8m^2 \tag{4.8}$$

$$\frac{\partial \theta}{\partial \beta}\frac{\partial \theta}{\partial \alpha} + \frac{\partial \phi}{\partial \alpha}\frac{\partial \phi}{\partial \beta} \sin^2\theta = 0 \tag{4.9}$$

$$(\frac{\partial \theta}{\partial \beta})^2 + (\frac{\partial \theta}{\partial \beta})^2 \sin^2\theta = 8m^2. \tag{4.10}$$

For solving equations (4.8) - (4.10) let us considers:

$$\theta = \theta(\beta),\ \phi = \phi(\alpha). \tag{4.11}$$

Equation (4.9) is satisfied automatically and the two equations (4.8) and (4.10) reduces to

$$(\frac{\partial \phi}{\partial \alpha})^2 \sin^2\theta = 8m^2 \tag{4.12}$$

$$(\frac{\partial \theta}{\partial \beta})^2 = 8m^2. \tag{4.13}$$

Equation (4.13) indicated that, the function $m$ is depended on $\beta$ only (i.e $m = m(\beta)$ ). Also equation (4.12) can be written as:

$$(\frac{\partial \phi}{\partial \alpha})^2 = \frac{8m^2}{\sin^2\theta} = K_1^2, \tag{4.14}$$

where $K_1$ is a constant. From equation (4.14) we get:

$$\phi = \varepsilon K_1 \alpha + K_2 \tag{4.15}$$

where $K_2$ is a constant of integration and $\varepsilon = \pm 1$.
From equation (4.13) we have

$$\frac{d\theta}{\sin\theta} = \varepsilon K_1 d\beta, \tag{4.16}$$

by integration we get:

$$\tan\frac{\theta}{2} = K_3 e^{\varepsilon K_1 \beta}, \qquad (4.17)$$

where $K_3$ is a constant of integration.

The function $m(\beta)$ can be determined from two equations (4.14) and (4.17) and take the form:

$$m(\beta) = \varepsilon K_1 \frac{K_3 e^{\varepsilon K_1 \beta}}{2\sqrt{2}\ K_3^2 e^{2\varepsilon K_1 \beta} + 1}, \qquad (4.18)$$

from equation (4.15) we can choose the constants such that $\phi = \alpha$, also equation (4.17) to be agree with the boundary of $\theta$ i.e $(-\pi < \theta < \pi)$ take the form:

$$\tan\frac{\theta}{2} = \varepsilon e^{\varepsilon \beta} \qquad (4.19)$$

The two equations (4.17) and (4.19) are determine the relation between the sky coordinate $(\theta, \phi)$ and the cone coordinate parametrization $(\beta, \alpha)$ .

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**مخروط الضوء في فضاء زمكاني لا يحقق مبدأ ماخ**

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تم دراسة هندسة مخروط الضوء في فضاء زمكانى لا يحقق مبدأ ماخ وذلك عند النقاط المفرده علي سطح المخروط ثم حُسبت اللامتغيرات التفاضليه من النوع الاول والثاني و دراسة سلوكها حول النقاط المفرده . أوجدنا العلاقه بين الإحداثيات الكونيه و الإحداثيات البارامتريه على مخروط الضوء في الفضاء الزمكاني الذى لا يحقق مبدأ ماخ.