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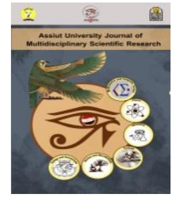

Supersymmetry as an Algebraic Approach to the Jaynes-Cumming Model with Stark Shift and Kerr-Like Medium

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ABSTRACT

According to of supersymmetry which states that in the standard model, each particle has a partner with a spin that differs by half of a unit. So fermions partners are bosons and vice versa. Supersymmetry has been introduced to many one-dimensional quantum systems, radial Shrödinger equation and for the known model Jaynes-Cummings model. Since bosonic and fermionic are so related to the photon of two level-system. of course using supersymmetry theory will be beneficial. Here, supersymmetry arguments are used to derive the wave function of Jaynes-Cummings model describes a two-level system (atom) interacts with a cavity field of single mode over a multi-photon transition where a Kerr-like medium and Stark term are considered. The supersymmetric operators related to the atomic system are introduced to diagonalize the Hamiltonian with a canonical transformation technique. Also, the eigenvalues spectrum and eigenvectors are exactly given. Then, alot of physics phenomena can be investigated and supersymmetry arguments can be applied on different optical systems.

INTRODUCTION

Styling and carrying out various photonic apparatuses widely depend on the principle of interference and propagation effects of optical waves. Then the major topic of the quantum optics is the light and matter interaction [1, 2]. Jaynes-Cumming model (JCM) is the main and most common model in quantum optics involves a two-level system (atom) in many field modes [3, 4]. It was solved exactly in the rotating wave approximation (RWA) and dipole. The integrable Hamiltonian which describes the JCM model was solved to get the system's eigenvalues and eigenvectors under many extensions and quantum effects. One of these extensions is the intensity-dependent coupling [5]. The other extension is a multi-level atom in state of two-level system. In particular three-level atoms with constant coupling [6-11], four-level atom has been discussed [12-24] and five-level atom in [25-33] where the nonlinear interactions are considered in various conditions. The expected quantum phenomena in the JCM are many. For instance, the collapse and revival the atomic population inversion, Rabi oscillations, [34], the dipole atom, photon antibunching [35], and many others.

Electric field affects the spectral lines of atoms and causes splitting and shifting phenomenon which is sometimes known as the Stark shift (or light shift) [36]. It is considered as an essential part of the field-atom interaction appearing from virtual transitions between atomic states. Also, the Stark effect of the Dicke Hamiltonian is studied when the fields are constant [37]. In the strong field case, the shift in energy levels is large compared to fine structures and is independent on electron spin. Then multiphoton transition is expounded accurately by Stark shift. That will be salient, if two levels interact with the middle level of an atomic system [38]. The detuning (depending on the intensity) in two-photon transitions is considered as Stark shift [39]. Moreover, the degree of quantum entanglement increases with the increasing of Stark shift parameter of two two-level atoms in a single-mode which interactes with Stark shift and degenerate two-photon transition [40].

Kerr effect is an electric field phenomenon and is exhibited by harmonic oscillator [41]. Kerr term and the Hamiltonian have similarity in the square of the number of photon operator. [42-44] discussed the non-linear interaction of a single mode cavity field with a Kerr medium and shows that interaction affects on the dynamic of an atomic population when the non-linear popularization of the JCM was considered. In the case of two multi mode cavity field, a two-level system was treated from the quantum dynamics view in the presence of a Kerr medium [45, 46]. The two-photon JCM model (non-degenerate case) was presented in [47] by using a method of operators when a Kerr like medium is considered. The entropies and the degree of entanglement in the mixed state for a multi-quanta JCM with Stark shift and Kerr- like medium effect was studied in [48].

The theoretical idea of supersymmetry, for each boson there must exist a supersymmetry fermion partner and vice-versa. In this paper, the dynamics of the atom-photon interaction is examined from another per-spective with JCM as a prototype example in the presence of Kerr-like medium and Stark shift. Since the two level-atom photon is a many-body system of bosonic and fermionic.Then, naturally, it is useful to use supersymmetry. Supersymmetry has been introduced to many one-dimensional quantum systems [49], radial Schrödinger equation [50] and for JCM [51, 53]. Hence, we will directly use this approach to calculate the eigenvalues and eigenfunctions of JCM and determine the wavefunction of the corresponding Hamiltonian. We consider a two-level atom interacting with a single mode cavity field through multi-poton transition when the Kerr-like medium and the Stark-shift are considered.

This paper is arranged as follows: In section 2, we present the atomic model under consideration. In section 3, the supersymmetric operators and their corresponding relations are given. In section 4, we give a simple realization of the diagonalized Hamiltonian of our model in the frame of supersymmetry by defining a suitable linear transformation in term of supersymmetry operators. In section 5, we provide the

Schrödinger equation solution of our Hamiltonian. Then section 6 is devoted to the conclusion.

THE HAMILTONIAN OF THE SYSTEM

In this section, we disply the model of a two-level system interacts with a cavity field of single mode wherea transition of multi-photon , Stark shift and the Kerr medium exist. The levels of energy are denoted by $|1>$ and $|2>$ where $|1>$ is the lower level and $|2>$ is the upper level. Under these considerations, the system's Hamiltonian (with $\hbar=1$) in RWA is written as

$$H=\omega_1\sigma_{11}+\omega_2\sigma_{22}+\Omega a^\dagger a+g(R\sigma_{12}+R^\dagger\sigma_{21})+a^\dagger a(\beta_1\sigma_{11}+\beta_2\sigma_{22})+\chi(a^\dagger)^2a^2 \quad (1)$$

where w is the atom transition frequency, $\sigma_{11}=|1><1|$ and $\sigma_{22}=|2><2|$ are the operators of the excited and ground state population respectively. $\sigma_{21}=|2><1|$ ($\sigma_{12}=|1><2|$) are the atomic lowering (raising) operators. The Fermionic atomic operators σ_{12}, σ_{21} and $\sigma_z=\sigma_{11}-\sigma_{22}$ are satisfying the commutation relations of the finite dimensional complex Lie algebra $sl(2,c)$, Ω is the cavity field frequency and $a, a^\dagger$ are the photon annihilation and creation operators of the cavity field which obey the bosonic field commutation relation $[a,a^\dagger]=1$, g is the atom-field coupling constant. The operators R and $R^\dagger$ are the following

$$R=a^kf(\hat{n}),\qquad R^\dagger=(a^\dagger)^kf(\hat{n}) \quad (2)$$

with a real valued function $f(\hat{n})$ of the operator $\hat{n}=a^\dagger a$ of the photon number and k is the photon number in the process of atom transition. These operators satisfy the following commutation

$$[R,\hat{n}]=kR,\qquad [R^\dagger,\hat{n}]=-kR^\dagger,$$

$$[R,R^\dagger]=\frac{(n+k)!}{n!}f^2(\hat{n}+k)-\frac{(n)!}{(n-k)!}f^2(\hat{n}).$$

Also, β_1 and β_2 are the Stark shift parameters (with respect to the two levels of the model) due to the virtual transitions to the intermediate level and they depend on the intensity. χ is the Kerr medium strength parameter of the quadratic nonlinearity modeling.

According to Heisenberg equation of motion and the former commutation relations we have the next constants of motion

$$I=\sigma_{11}+\sigma_{22},\qquad N'=a^\dagger a-k\sigma_{22} \quad (3)$$

where I is the unit operator and N' is the number of excitation. Then equation of Hamiltonian (1) becomes in the form

$$H=\left(\omega_1-\frac{\Delta}{2}\right)I+\frac{\Delta}{2}\sigma_z+\Omega N+g(R\sigma_{12}+R^\dagger\sigma_{21})+a^\dagger a\left(\frac{\beta}{2}+\frac{\beta'}{2}\sigma_z\right)+\chi(a^\dagger)^2a^2 \quad (4)$$

where $\Delta = \omega_2 - \omega_1 - k\Omega$ is the detuning parameter, and $\beta = \beta_1 + \beta_2$, $\beta' = \beta_1 - \beta_2$.

SUPERSYMMETRIC APPROACH

In this section we introduce the supersymmertic generators of our model. We define them in the following three operators

$$E \coloneqq R\sigma_{12} = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}, \qquad F \coloneqq R^{\dagger}\sigma_{21} = \begin{pmatrix} 0 & 0 \\ R^{\dagger} & 0 \end{pmatrix}$$

$$G := \{F, E\} = \begin{pmatrix} RR^{\dagger} & 0 \\ 0 & R^{\dagger}R \end{pmatrix} \tag{5}$$

which satisfy the following anticommutators

$$\{F, F\} = \{E, E\} = 0, \quad \{F, E\} = \{E, F\} = 0,$$

$$\{E, \sigma_z\} = \{F, \sigma_z\} = 0, \quad \{E, G\} = \{F, G\} = 0. \tag{6}$$

The operators F, E, and G form an algebra called supersymmetic Lie algebra and satisfy the following relations

$$[G, F] = [G, E] = [G, \sigma_z] = 0, \qquad [E, F] = G\sigma_z,$$

$$[E, \sigma_z] = -2E, \qquad [F, \sigma_z] = 2F, \qquad \sigma_z(E - F) = (E + F) \tag{7}$$

where [ ] is the Lie bracket.

DIAGONALIZATION OF THE HAMILTONIAN

We use the relations (5)-(7) to rewrite the Hamiltonian (4) in term of the supersymmetric generators F, E and N

$$H = (\omega_1 - \frac{\Delta}{2})I + \Omega G + \frac{\Delta}{2}\sigma_z + a^{\dagger}a(\frac{\beta}{2} + \frac{\beta'}{2}\sigma_z) + \chi {a^{\dagger}}^2 a^2 + g(F + E). \tag{8}$$

Let’s introduce the following linear transformation in term of the supersymmetric generators F, and E

$$T = \exp(\frac{-\theta}{2M}(E - F)) \tag{9}$$

where

$$M = \begin{pmatrix} \frac{1}{\sqrt{RR^{\dagger}}} & 0 \\ 0 & \frac{1}{\sqrt{R^{\dagger}R}} \end{pmatrix}. \tag{10}$$

By using the relations (6) and (7), the transformations U and U^{-1} read as

$$U = \cos\frac{\theta}{2} - \frac{1}{M}(E-F)\sin\frac{\theta}{2}, \qquad U^{-1} = \cos\frac{\theta}{2} + \frac{1}{M}(E-F)\sin\frac{\theta}{2}. \tag{11}$$

Also we can calculate the following relations

$$U^{-1}\sigma_z U = \sigma_z\cos\theta - \frac{1}{M}(E+F)\sin\theta \tag{12}$$

$$U^{-1}a^\dagger a U = a^\dagger a + k\sigma_z\sin^2\frac{\theta}{2} + \frac{k}{2\sqrt{G}}(E+F)\sin\theta$$

$$U^{-1}(a^\dagger a)^2 U = (U^{-1}a^\dagger a U)(U^{-1}a^\dagger a U). \tag{13}$$

Using the transformation U the Hamiltonian (4) becomes

$H = U^{-1}HU$

$$= (\omega_1 - \frac{\Delta}{2} + \frac{\chi k^2}{4} - \frac{\beta' k}{4})I + \Omega G + \chi A(A-1) + \frac{\beta}{2}A - \sqrt{B^2 + g^2 G}\sigma_z \tag{14}$$

where $A = a^\dagger a + \frac{k}{2}\sigma_z$, $B = \frac{\chi k}{2}(2A-1) - \frac{1}{2}(\beta A - \frac{\beta' k}{2}) - \frac{\Delta}{2}$ and $g\cos\theta = -\sin\frac{B}{\sqrt{N}}$.

THE WAVE FUNCTION OF THE SYSTEM

We suppose the eigenstates of H of the considered model by

$$|\psi_1'\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2'\rangle = \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix}. \tag{15}$$

Theses eigenstates satisfy the following relations under the action of the operators N, A, G, and σ_z

$$N|\psi_1'\rangle = n|\psi_1'\rangle, \qquad N|\psi_2'\rangle = n|\psi_2'\rangle,$$

$$A|\psi_1'\rangle = (n+\frac{k}{2})|\psi_1'\rangle, \qquad A|\psi_2'\rangle = (n+\frac{k}{2})|\psi_2'\rangle,$$

$$G|\psi_1'\rangle = \frac{(n+k)!}{n!}f^2(n+k)|\psi_1'\rangle, \qquad G|\psi_2'\rangle = \frac{(n+k)!}{n!}f^2(n+k)|\psi_2'\rangle,$$

$$\sigma_z|\psi_1'\rangle = |\psi_1'\rangle, \qquad \sigma_z|\psi_2'\rangle = -|\psi_2'\rangle, \tag{16}$$

with

$$A|n,1\rangle = (n+\frac{k}{2})|n,1\rangle, \qquad A|n+k,2\rangle = (n+\frac{k}{2})|n+k,2\rangle,$$

$$B|n,1\rangle = [\frac{\chi k}{2}(2n+k-1) - \frac{1}{2}(\beta_1 n + \beta_2 n + \beta_2 k) - \frac{\Delta}{2}]|n,1\rangle,$$

$$B|n+k,2\rangle = [\frac{\chi k}{2}(n+k-1) - \frac{1}{2}[(\beta_1+\beta_2)(n+\frac{k}{2}) - \frac{(\beta_1-\beta_2)k}{2}] - \frac{\Delta}{2}]|n+k,2\rangle.$$

The eigenvalues of the diagonalized Hamiltonian H (14) is given by

$$E_1 = \delta + \sqrt{\xi^2+\eta}, \qquad E_2 = \delta - \sqrt{\xi^2+\eta}, \tag{17}$$

where:

$$\delta = \omega_1 - \frac{\Delta}{2} + \frac{\chi k^2}{4} - \frac{(\beta_1 - \beta_2)k}{4} + \Omega n + \chi(n + \frac{k}{2})(n + \frac{k}{2} - 1) + \frac{(\beta_1 + \beta_2)}{2}(n + \frac{k}{2}),$$

$$\xi = \frac{\chi k}{2}(2n + k - 1) - \frac{1}{2}(\beta_1 n + \beta_2 n + \beta_2 k) - \frac{\Delta}{2},$$

$$\eta = g^2 \frac{(n+k)!}{n!} f^2(n + k). \quad (18)$$

It’s crucial to mention that there is another eigenstate $|\psi_3 >= \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix}$, $(n \leqslant k - 1)$ to cover all possible n. We can see that

$$H|\psi_3 >= E_3|\psi_3 >, \quad (19)$$

with

$$E_3 = \omega_1 - \Delta + +\Omega(n - k) - \frac{(\beta_1 - \beta_2)k}{4} + \chi(n^2 - n + \frac{k^2}{2}). \quad (20)$$

The eigenstates of the original Hamiltonian H are given by

$$|\psi_1 >= U|`\psi_1 >= \cos\frac{\theta}{2}|\psi_1{}' > +\sin\frac{\theta}{2}|\psi_2{}' >$$

$$|\psi_2 >= U|`\psi_2 >= -\sin\frac{\theta}{2}|\psi_1{}' > +\cos\frac{\theta}{2}|n + k >$$

$$|\psi_3 >= U|\psi_3{}' >= \cos\frac{\theta}{2}|\psi_3{}' > \quad (21)$$

To find the wave function $|\psi(t) >$ of the system, we consider the system is initially in an arbitrary state $|\psi(0) >$

$$|\psi(0) >= \sum_{m=0}^{\infty} q_{m+k}|m + k, 2\rangle \quad (22)$$

and the Schrödinger equation $i\frac{d}{dt}|\psi(t) >= H|\psi(t) >$ becomes

$$\left|\psi(t) >= \sum_{n=0}^{\infty} e^{-iHt}\right|\psi(0) > = \sum_{n=0}^{\infty} e^{-iHt}[\alpha_n|\psi_1 > +\beta_n|\psi_2 > +\gamma_n|\psi_3 >], \quad (23)$$

where

$$\alpha_n =< \psi_1|\psi(0) >, \qquad \beta_n =< \psi_2|\psi(0) >, \qquad \gamma_n =< \psi_3|\psi(0) >, \quad (24)$$

and $\sum_{n=0}^{\infty} < \psi_1|\psi_1 > +< \psi_2|\psi_2 > + \sum_{n=0}^{k-1} < \psi_3|\psi_3 >= 1$.

Then

$$|\psi(t) >= \sum_{n=0}^{\infty} e^{-i\delta t} q_{n+k}\left[-ig\sqrt{\frac{(n + k)!}{n!}}\frac{\sin\lambda_n t}{\lambda_n}\middle|n, 1 > + \left(\cos\lambda_n t - i\frac{\Delta\sin\lambda_n t}{2\lambda_n}\right)\middle| n, 2 >\right]$$

$$+ \sum_{n=0}^{k-1} e^{-iE_3 t} q_n|n, 2 > \quad (25)$$

where $\lambda_n = \sqrt{\xi^2 + \eta}$.

CONCLUSION

We have studied the two-level JCM model with Stark shift and Kerr-like medium in details. Since of the canonical relations between bosns and fermions, we detected that the supersymmetry approach can be used in the considered atomic model. We construct the supersymmetric operators and introduce an exact transformation to get the diagonal Hamiltonian of the corresponding model. The eigenvalues, eigenvectors and the time evaluation wave function of the system are introduced. Now, a lot of physics phenomena can be investigated.

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