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Some New Results of Revolution Surfaces in Euclidean 3-Space E^3

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ABSTRACT

In this paper, a new technique to study some characteristic properties of revolution surfaces in Euclidean 3-space E^3. We construct and obtain the necessary conditions of Weingarten, linear Weingarten, bi-conservative, harmonic, bi-harmonic and stability revolution surfaces in E^3. Using computer-aided geometric design, we present and plot many applications.

INTRODUCTION

Besides the curves, rotational surfaces were among the earliest topics in differential geometry to be addressed. In both engineering and science, the utilization of surfaces of revolution is crucial. Because they occur frequently in nature, surfaces of revolution have long been recognized as both common and well-known in geometric modeling. For instance, in mathematics, human artifacts, and technological practice. In addition, several items from daily life, including cans, furniture legs, and table glasses. They serve as illustrations of revolution surfaces. Additionally, the simple act of turning wood creates surfaces that are in a state of revolution [1, 2].

Weingarten first introduced W-surfaces, also known as Weingarten surfaces, in 1861 in relation to the challenge of identifying all surfaces that are isometric to a given surface of revolution. They have attracted geometers' interest over time. W-surface applications for computer-aided design and shape analysis are shown in [3]. In Euclidean 3-space, a linear Weingarten surface, also known as a LW-surface, is a surface whose mean curvature H and Gaussian curvature G meet the relation $aH + bG = c$, where $a, b, c \in R$. Numerous geometers tried to find examples of LW-surfaces for a very long period; for instance, see [4]. In the general case, the classification of LW-surfaces is now mostly undefined. They have historically been of interest to geometers, especially when the surface is closed, as shown in [5-10].

One of the most attractive geometric objects and ones that have significant physical significance are harmonic surfaces. These are surfaces in space that localise area minimization, meaning that any sufficiently small section of the surface has the smallest area among all surfaces sharing the same boundary. In the real world, they manifest themselves spontaneously. According to physical laws, a soap film that is spanned by a specific boundary curve will take on the appearance of a harmonic surface. There are numerous applications for these surfaces. Any Riemannian manifold of at least three dimensions, or a manifold with a smooth field of inner products on their tangent spaces, is used to study them. In complex Euclidean spaces C^n for $n > 1$ are rather special examples harmonic surfaces [11].

The study of bi-conservative and bi-harmonic surfaces is nowadays a very active research subject. Many enjoyable results of these types have been obtained in the last decade. In the last few years, from the theory of bi-harmonic submanifolds, arose the study of bi-conservative submanifolds that enforced themselves as a very hopeful and interesting research topic. Closely related to the theory of bi-harmonic submanifolds, the study of bi-conservative submanifolds is a very recent and delectable topic in the field of differential geometry [12].

The differential geometry of stability issues involving generic surfaces has recently piqued the curiosity of several geometers. As more researchers became involved and saw outcomes over the past few decades, this interest grew quickly. One may specifically mention [13-18]'s works. The interaction of classical differential geometry with the calculus of stability is one of its most fascinating and significant features. The theory of harmonic surfaces, for instance, is where the roots of the calculus of stability can be found. Recent years have seen the careful study of a seemingly novel sort of stability problem proposed by the stability principles that give rise to the general theory of relativity's field equations. One is, at least implicitly, concerned with a multiple integral in the calculus of stability in the case of the earlier applications [15, 19].

In this paper, we studied the possibility of obtaining the necessary conditions for revolution surfaces to become type L/W-surfaces, bi-conservative, harmonic, bi-harmonic and stable in Euclidean 3-Space E^3. In the last section, we were able to solve the previous equations by giving special cases and using a new method. We got theoretical or numerical solutions to those equations. Then, we translated these results into geometric shapes.

2. Geometric preliminaries

In this section, we introduce some basic definitions and relations for our analysis for the surfaces in Euclidean 3-space [11, 17, 20-26].

Definition 2.1 [20, 21] We say M a revolution surface which is generated by a plane curve r(u) when it is rotated around a straight line in the same plane.The parametrization of the plane curve is given by

$$r(u) = (\phi(u), \psi(u)). \tag{2.1}$$

Then the parametrization of revolution surface is given by

$$M: X(u,v) = (\phi(u)\cos v,\ \phi(u)\sin v,\ \psi(u)), 0 < v < 2\pi,\ \phi(u) \neq 0. \tag{2.2}$$

The unit normal vector field of M can be defined by

$$N = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}, \quad X_i = \frac{\partial X}{\partial u_i}\ ,\ u_1 = u,\ u_2 = v. \tag{2.3}$$

The first fundamental form I of the surface M is given by

$$I = \sum_{i,j=1}^{2} g_{ij}\, du^i\, dv^j. \tag{2.4}$$

With the coefficients

$$g_{ij} = \langle X_i, X_j \rangle. \tag{2.5}$$

The discriminate g of the first fundamental quadratic form is

$$g = det(g_{ij}) = g_{11}g_{22} - g_{12}^2. \tag{2.6}$$

The second fundamental form II of the surface M is given by

$$II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, dv^j. \tag{2.7}$$

With the coefficients

$$h_{ij} = \langle X_{ij}, N \rangle,\ X_{ij} = \frac{\partial^2 X}{\partial u_i\, \partial u_j}. \tag{2.8}$$

The discriminate h of the second fundamental quadratic form is

$$h = det(h_{ij}) = h_{11}h_{22} - h_{12}^2. \tag{2.9}$$

Under this parametrization of the surface M, the Gaussian curvature and the mean curvature respectively, are given by

$$G = k_1 k_2 = \frac{h}{g} \quad and \quad H = \frac{k_1 + k_2}{2} = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij}\, h_{ij}, \tag{2.10}$$

where the principal curvatures at a point p on the surface M, denoted

by k_1 and k_2 are the global maximum and the global minimum of the sectional

curvature at the point p. And g^{ij} denotes the associated matrix with its inverse

(g_{ij}). i.e, $g^{ij} = (g_{ij})^{-1}$.

Definition 2.2 [26] W-surfaces are the surfaces which satisfying $\eta(G,H) = 0$,

or, the corresponding Jacobian determinant is identically zero, i.e,

$$\eta(G,H) = \left|\frac{\partial(G,H)}{\partial(u,v)}\right| \equiv 0. \tag{2.11}$$

We can rewrite the condition (2.11) as follow

$$G_u H_v - G_v H_u = 0. \tag{2.12}$$

Definition 2.3 [22, 23] LW-surfaces are the surfaces which satisfying the linear equation

$$aG + bH = c, (a,b,c) \neq (0,0,0) \in R. \tag{2.13}$$

When the constant b=0, a LW-surface reduces to a surface with constant Gaussian curvature. And when the constant a =0, a LW-surface reduces to a surface with constant mean curvature. In such a sense, the LW-surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature.

Definition 2.4 [24]. A surface M in Euclidean 3-space is bi-conservative if the mean curvature function H satisfies

$$A\,(grad\,H) = -H\; grad\,H. \tag{2.14}$$

This condition can be split into two differential equations as follows

$$a_{11}H_u + a_{12}H_v + HH_u = 0, \tag{2.15}$$

$$a_{21}H_u + a_{22}H_v + HH_v = 0, \tag{2.16}$$

where $A = (a_{ij}),\ i,j = 1,2$ is given by

$$\left.\begin{array}{l} a_{11} = (h_{11}g_{22} - h_{12}g_{12})/g, \; a_{12} = (h_{12}g_{11} - h_{11}g_{12})/g \\ a_{21} = (h_{12}g_{22} - h_{22}g_{12})/g, \; a_{22} = (h_{22}g_{11} - h_{12}g_{12})/g \end{array}\right\} \quad (2.17)$$

Definition 2.5 [11]. A smooth surface in E^3 is a harmonic surface (minimal surface) if its mean curvature equals zero at every point, i.e, $k_1 + k_2 = 0$, where k_1 and k_2 are the principal curvatures.

Definition 2.6 [25]. A surface M in Euclidean 3-space is said to be bi-harmonic if it satisfies the equation $\Delta^2 X = 0$, where $X = X(u, v)$ is the vector function representation of the surface M.

According to the well-known Betrami's formula $\Delta X = -2\vec{H}$, the bi-harmonic condition in E^3 is also known as the equation

$$\Delta\vec{H} = 0, \quad \vec{H} = H\overrightarrow{N}. \quad (2.18)$$

Where Δ is the Laplacian operator (Laplacian-Beitrami operator) with respect to

the first fundamental form of X and is given by

$$\Delta = -\frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial u^i}\left[\sqrt{g}\, g^{ij}\frac{\partial}{\partial u^j}\right], \quad (2.19)$$

where, (g^{ij}) denotes the associated matrix with its inverse (g_{ij}). i.e, $(g^{ij}) = (g_{ij})^{-1}$ and $g = det(g_{ij})$ and u^i *are the local coordinate on M*.

Definition 2.7 [17]. The oriented compact immersion $X: M \to E^3$ is stable

with respect to the integral $\int_M H^2 dA$ iff the following condition is valid

$$\Delta H + 2H(H^2 - G) = 0, \quad (2.20)$$

where dA is the volume element of M and Δ is given by (2.19).

3. GENERAL PROPERTIES OF REVOLUTION SURFACE M

In this section, we shall describe and derive the fundamental quantities of revolution surface M. Some properties of this surface are introduced. The general conditions for this surface to become of type L/w-surface, bi-conservative, harmonic, bi-harmonic and stable are derived.

From (2.2) the coefficients of the first fundamental forms of the surface M are given by

$$g_{11} = \psi'^2 + \phi'^2, \quad g_{22} = \phi^2, \quad g_{12} = 0. \tag{3.1}$$

It is convenient to assume that the rotating curve is parameterized by arc length, that is, that

$$\phi'^2 + \psi'^2 = 1, \tag{3.2}$$

and ϕ is always positive, it follows that the parabolic points are given by either $\psi' = 0$ (the tangent line to the generator curve is perpendicular to the axis of rotation) or $\phi'\psi'' - \psi'\phi'' = 0$ (the curvature of the generator curve is zero). A point which satisfies both conditions is a planar point, since these conditions imply that $h_{11} = h_{12} = h_{22}$.

By differentiating (3.2) we obtain $\phi'\phi'' = -\psi'\psi''$. Therefore, the metric of the first fundamental form g of the surface (2.2) is given by

$$g = \phi^2. \tag{3.3}$$

The unit normal vector field N is given by

$$N = (-\psi' \cos v, -\psi' \sin v, \phi'). \tag{3.4}$$

Consequently, the coefficients of the second fundamental form are written as follows:

$$h_{11} = \phi'\psi'' - \psi'\phi'', \quad h_{22} = \phi \;\; \psi', \quad h_{12} = 0. \tag{3.5}$$

Hence the metric of the second fundamental form h is given by

$$h = \phi \, \psi'(\phi'\psi'' - \psi'\phi''). \tag{3.6}$$

Thus, and using (3.2), the Gaussian curvature is given by

$$G = -\frac{\psi'(\psi'\phi'' - \phi'\psi'')}{\phi} = -\frac{\psi'^2\phi'' + \phi'^2\phi''}{\phi} = -\frac{\phi''}{\phi}\left(\psi'^2 + \phi'^2\right) = -\frac{\phi''}{\phi}. \tag{3.7}$$

The principal curvatures of a surface of revolution are given by

$$k_1 = \phi'\psi'' - \psi'\phi'', \; k_2 = \frac{\psi'}{\phi}, \tag{3.8}$$

hence, the mean curvature of a such surface is

$$H = \frac{\psi' + \phi(\phi'\psi'' - \psi'\phi'')}{2\phi}, \tag{3.9}$$

and a_{ij} are given by

$$a_{11} = (\phi'\psi'' - \psi'\phi''),\ a_{22} = \frac{\psi'}{\phi},\ a_{12} = 0,\ a_{21} = 0. \quad (3.10)$$

From (2.2) if this is taken as axis of z and u denotes perpendicular distance from it, the parametrization of the surface M is given by

$$M_0 : X(u,v) = (u\cos v,\ u\sin v,\ \psi(u)). \quad (3.11)$$

Corollary 3.1 The Gaussian and mean curvature G and H of surface (3.11) are given by

$$G = 0,\ H = \frac{1}{2}(u^{-1}\psi' + \psi''), \quad (3.12)$$

and a_{ij} are given by

$$a_{11} = \psi'',\ a_{22} = u^{-1}\psi',\ a_{12} = 0,\ a_{21} = 0. \quad (3.13)$$

From the foregoing results, one can get the following:

Lemma 3.1 The coordinate patch of M is orthogonal $(g_{12} = 0)$.

Lemma 3.2 The coordinate patch of M is principal$(g_{12} = h_{12} = 0)$.

Lemma 3.3 The coordinate patch of M is of type asymptotic orthogonal $(h_{11} = h_{22} = 0, g_{12} = 0)$.

Lemma 3.4 U-clairaut patch $(g_{11,2} = g_{22,2} = g_{12} = 0)$, $g_{ij,k} = \frac{\partial g_{ij}}{\partial u_k}$.

Lemma 3.5 V-clairaut patch $\left(g_{11,1} = g_{22,1} = g_{12} = 0\right)$.

Remark 3.1 We note that the conditions in lemmas (3.1, 3.2 and 3.4) are vanished identically. While the conditions in lemmas (3.3 and 3.5) are valid for a constant function $\psi(u)$.

Since $H_v = 0,\ G_v = 0$, we have:

Corollary 3.2 The surface M_0 is W-surface.

Corollary 3.3 The revolution surface M_0 is LW-surface if the following equation is valid

$$m_1(u^{-1}\psi' + \psi'') - 2m_2 = 0, \quad (3.14)$$

where m_1 and m_2 are constants.

Theorem 3.1 The revolution surface M_0 is bi-conservative if the following two equations are valid

$$a_{11}H_u + a_{12}H_v + HH_u = (\psi'' + \tfrac{1}{2}(u^{-1}\psi' + \psi''))(u^{-1}\psi'' - u^{-2}\psi' + \psi''') = 0, \tag{3.15}$$

$$a_{21}H_u + a_{22}H_v + HH_v = 0, \tag{3.16}$$

where H_u, H_v *and* a_{ij} are given by

$$\left.\begin{aligned} H_u &= \tfrac{1}{2}(u^{-1}\psi'' - u^{-2}\psi' + \psi'''),\ H_v = 0 \\ a_{11} &= \psi'',\ \ a_{22} = u^{-1}\psi', a_{12} = 0, a_{21} = 0 \end{aligned}\right\} \tag{3.17}$$

Since $H_v = a_{21} = 0$, we have:

Remark 3.1 The condition (3.16) is vanished identically.

Corollary 3.4 The surface M_0 is harmonic if the following equation is valid

$$\psi' + u\,\psi'' = 0. \tag{3.18}$$

This equation is nonlinear ODE of second order and its solution as follow

$$\psi(u) = c_1 \ln u + c_2, \tag{3.19}$$

where c_1 and c_2 are two arbitrary constants. Thus, we have:

Corollary 3.5 The surface M is harmonic if the function $\psi(u)$ has the form (3.19).

Theorem 3.2 The revolution surface M_0 is bi-harmonic if the following conditions are valid

$$a_1 = w_1 cos\, v = 0,\ \ a_2 = w_1 sin\, v = 0, \tag{3.20}$$

$$a_3 = 2u^{-1}\psi^{(3)} - u^{-2}\psi'' + u^{-3}\psi' + \psi^{(4)} = 0, \tag{3.21}$$

where w_1 is given by

$$w_1 = 3u^{-1}\psi''^2 - 3u^{-2}\psi'\psi'' + \psi'\psi^{(4)} + 3u^{-1}\psi'\psi^{(3)} + 3\psi''\psi^{(3)}. \tag{3.22}$$

Theorem 3.3 The revolution surface M_0 is stable iff the following condition is valid

$$2(2u^{-1}\psi^{(3)} - u^{-2}\psi'' + u^{-3}\psi' + \psi^{(4)}) - (u^{-1}\psi' + \psi'')^3 = 0. \tag{3.23}$$

4. APPLICATIONS ON REVOLUTION SURFACE M_0

In the previous section, several of nonlinear ODEs were appeared. Thus their general solutions are much more complicated and can only be solved in special cases. Since the cases where these equations can be explicitly integrated are rare, numerical solutions of these equations are the only way to get with the previous conditions. That is why in this section, a new technique is used, hence we have investigated some of special analytical and numerical solutions using Matlab program v. 18.

Here, we give the following cases:

4.1. Case 1. If we put $\psi(u) = u + 1$, we denote this surface by M_1 (see Fig. 1).
So using Eqs. (3.14 - 3.22), we have the following corollaries:

(i) M_1 is Lw-surface if $u = m_1/2m_2$.
(ii) M_1 is not bi-conservative because the condition (3.15) equal $\frac{-1}{2u^3} \neq 0$.
(iii) M_1 is not harmonic because the condition (3.18) equal $1 \neq 0$.
(iv) M_1 is not bi-harmonic because the condition (3.20) equal $u^{-3} \neq 0$.
(v) M_1 is unstable because the condition (3.22) equal $u^{-3} \neq 0$.

4.2. Case 2. If we put $\psi(u) = u^2 + u + 1$, we denote this surface by M_2 (see Fig. 2).
Based on the Eqs. (3.14-3.22), we have the following corollaries:
(i) M_2 is Lw-surface if the following equation is valid

$$(4u + 1)m_1 - 2um_2 = 0. \tag{4.1}$$

If we take m_1=3, m_2=4, say, we get $u = -3/4$.

(ii) M_2 is bi-conservative if $u = -1/8$.
(iii) M_2 is harmonic if $u = -1/4$.
(iv) M_2 is not bi-harmonic because the equation (3.20) equal $u^{-3} \neq 0$.
(v) M_2 is stable iff the following equation is valid

$$64u^3 + 48u^2 + 12u - 1 = 0, \tag{4.2}$$

and the real solution of this equation is given by $u = \frac{1}{4}\left(\sqrt[3]{2} - 1\right)$,

and the other two roots of Eq. (4.2) are complex.
The Eq. (4.2) and its real solution are illustrated in (Fig. 3).

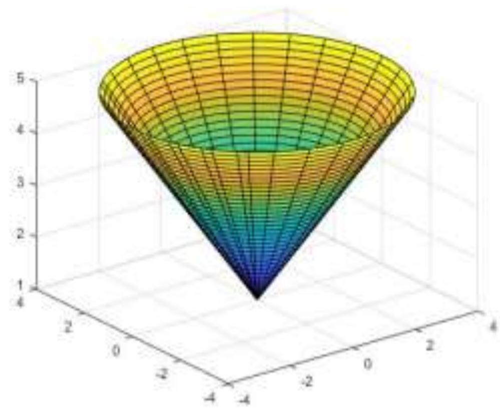

Figure 1: $Graph\ \ of\ \ M_1$

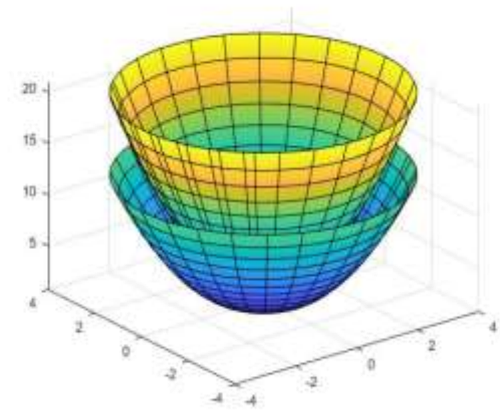

Figure 2: $Graph\ \ of\ \ M_2$

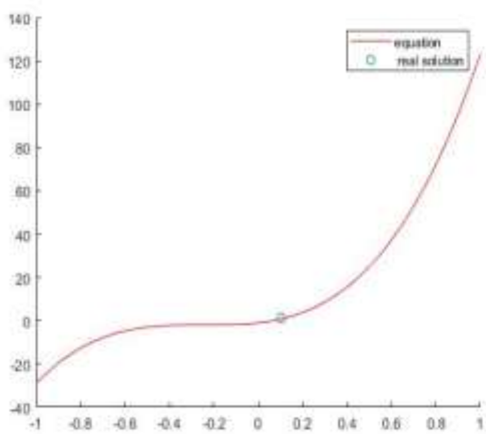

Figure 3: Stability of M_2

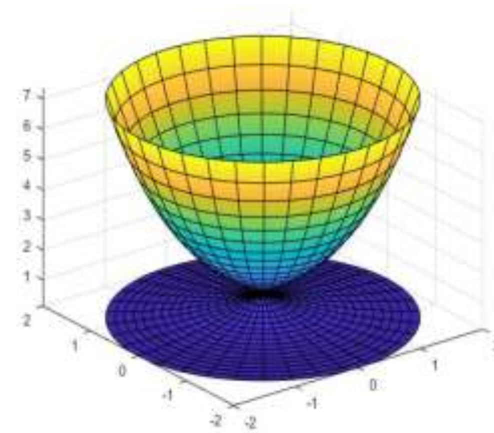

Figure 4: Graph of M_3

4.3. Case 3. If we put $\psi(u) = e^u$, we denote this surface by M_3 (see Fig. 4). Similarly, using (3.14-3.22), we have the following corollaries:

(i) M_3 is Lw-surface if the following equation is valid

$$m_1 e^u(u^{-1} + 1) - 2m_2 = 0. \tag{4.3}$$

The Eq. (4.3) is illustrated in (Fig. 5).

(ii) M_3 is bi-conservative if the following equation is valid

$$3u^3 + 4u^2 - 2u - 1 = 0. \tag{4.4}$$

The solution of this equation is given by

$$\left.\begin{array}{l} u_1 = -1/3 \\ u_{2,3} = \frac{1}{2}(-1 \pm \sqrt{5}) \end{array}\right\} \tag{4.5}$$

The Eq. (4.4) and its solution are illustrated in (Fig. 6).

(iii) M_3 is harmonic if $u = -1$.
(iv) M_3 is bi-harmonic if the following polynomials are valid

$$a_1 = w_3 \ \cos v = 0, \quad a_2 = w_3 \ \sin v = 0 \tag{4.6}$$

$$a_3 = u^3 + 2u^2 - u + 1 = 0, \tag{4.7}$$

where w_3 is given by

$$w_3 = 4u^2 + 6u - 3. \tag{4.8}$$

The Eqs. (4.6) and (4.7) are illustrated in (Fig. 7).

(v) M_3 is stable iff the following polynomial is valid

$$(u+1)^3 e^{2u} - 2u^3 - 4u^2 + 2u - 2 = 0. \tag{4.9}$$

The numerical solution of this equation is $u = 456/3685$.

The Eq. (4.9) and its numerical solution are illustrated in (Fig. 8).

4.4. Case 4. If we put $\psi(u) = \log u$, we denote this surface by M_4 (see Fig. 9). Analogously, from Eqs. (3.14-3.22), we have the following corollaries:
(i) M_4 is not Lw-surface because the Eq.(3.14) equal $-8 \neq 0$.
if $m_1 = 3, \ m_2 = 4$.
(ii) M_4 is bi-conservative because the equation (3.15) is valid.
(iii) M_4 is harmonic because the equation (3.18) is valid.
(iv) M_4 is bi-harmonic because the conditions (3.19) and (3.20) are valid.
(v) M_4 is stable because the condition (3.22) is valid.

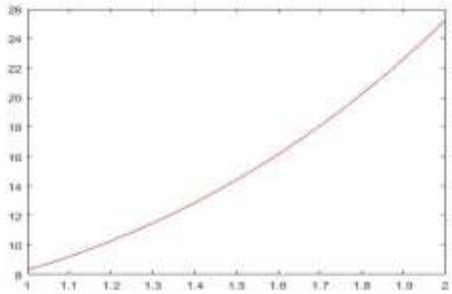

Figure 5: Lw-surface of M_3

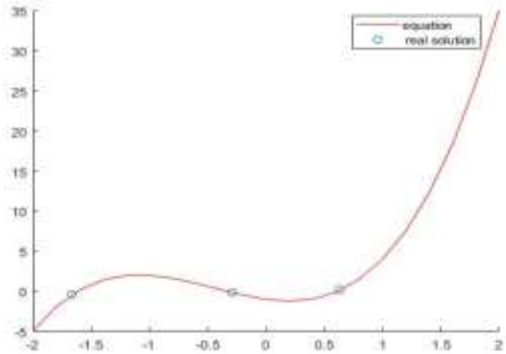

Figure 6: Bi-conservative of M_3

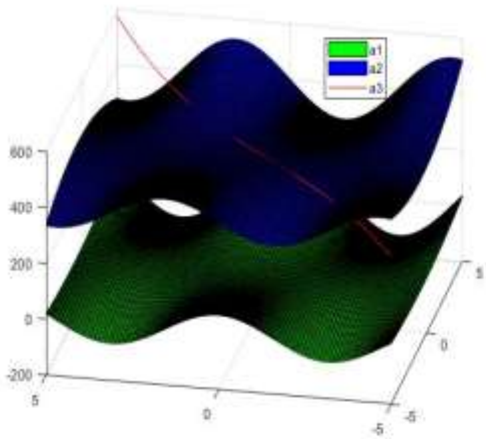

Figure 7: Bi-harmonic of M_3

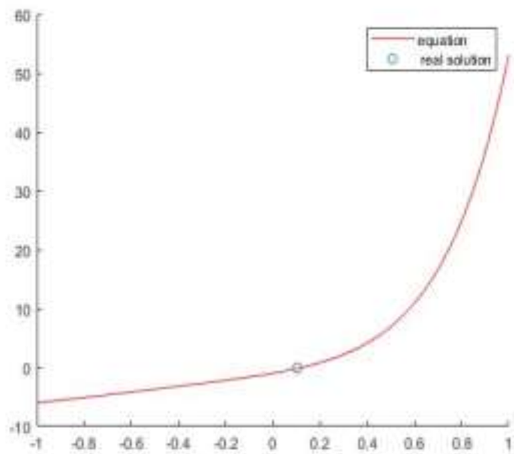

Figure 8: Stability of M_3

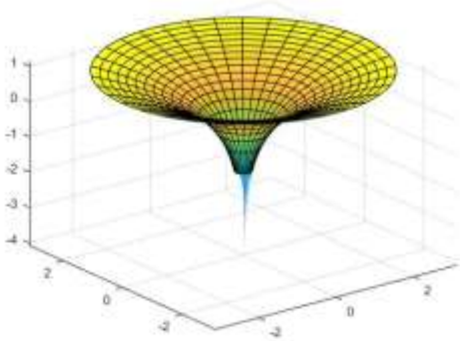

Figure 9: Graph of M_4

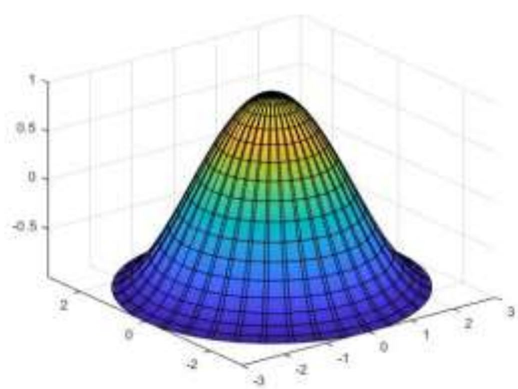

Figure 10: Graph of M_5

4.5. Case 5. If we put $\psi(u) = cos\ u$, we denote this surface by M_5 (see Fig. 10).

Taking into account the Eqs. (3.14-3.22), we have the following corollaries:

(i) M_5 is Lw-surface if the following equation is valid

$$m_1(sin\ u + u\ cos\ u) + 2um_2 = 0. \tag{4.10}$$

The solution of this equation is $u = 0$.

The equation (4.10) and its solution are illustrated in (Fig. 11).

(ii) M_5 is bi-conservative if the following equation is valid

$$(3cos\ u + u^{-1}sin\ u)(u^{-1}cos\ u - (u^{-2} + 1)sin\ u) = 0. \tag{4.11}$$

The numerical solution of this eqation is $u = -28809/131$.

The Eq. (4.11) and its numerical solution are illustrated in (Fig. 12).

(iii) M_5 is harmonic if the following equation is valid .

$$cos\ u + u^{-1}sin\ u = 0. \tag{4.12}$$

The numerical solution of this equation is $u = -28699/126$.

The Eq. (4.12) and its numerical solution are illustrated in (Fig. 13).

(iv) M_5 is bi-harmonic if the following polynomials are valid

$$a_1 = w_5\ cos\ v = 0,\ \ a_2 = w_5\ sin\ v = 0, \tag{4.13}$$

$$a_3 = (2u^2 - 1)\ sin\ u + u(u^2 + 1)\ cos\ u = 0, \tag{4.14}$$

where w_5 is given by

$$w_5 = 3u^{-1}(cos^2 u - sin^2 u) - (3u^{-2} + 4)sin\ u\ \ cos\ u. \tag{4.15}$$

The Eqs. (4.13) and (4.14) are illustrated in (Fig. 14).

(v) M_5 is stable iff the following equation is valid

$$u^3(cos\ u + u^{-1}sin\ u)^3 + 2u^3cos\ u + 2ucos\ u + 4u^2sin\ u - 2sin\ u = 0. \tag{4.16}$$

The numerical solution of this equation is $u = -7061/31$.

The Eq. (4.16) and its numerical solution are illustrated in (Fig. 15).

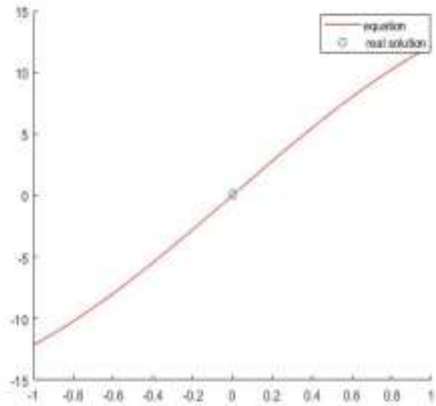

Figure 11: Lw-surface of M_5

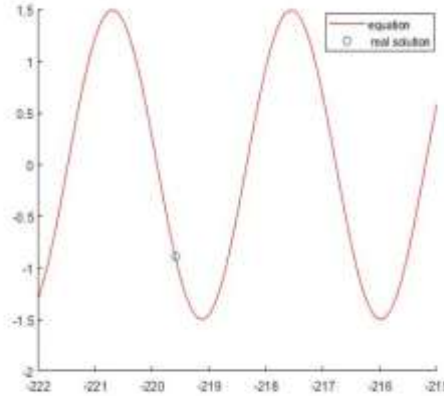

Figure 12: Bi-conservative of M_5

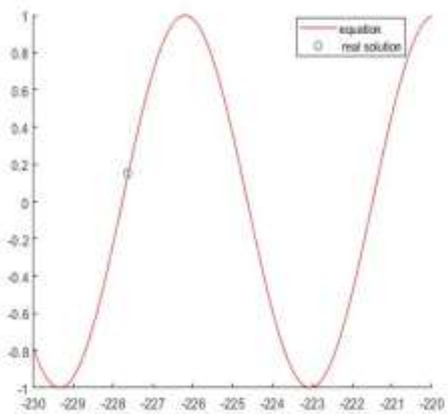

Figure 13: Harmonic of M_5

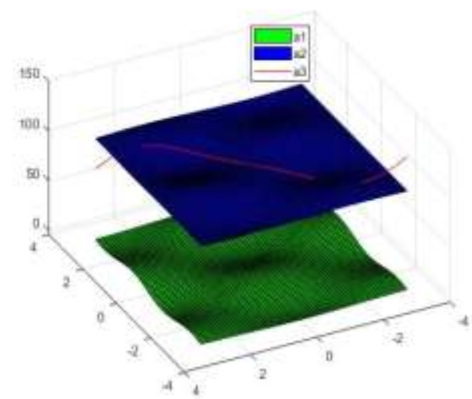

Figure 14: Bi-harmonic of M_5

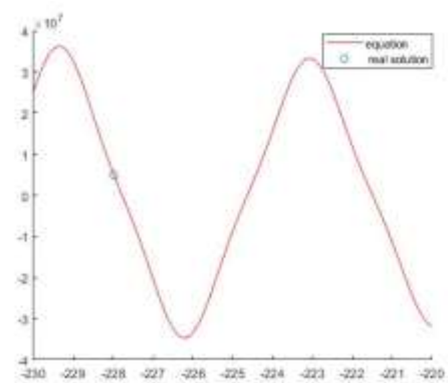

Figure 15: Stability of M_5

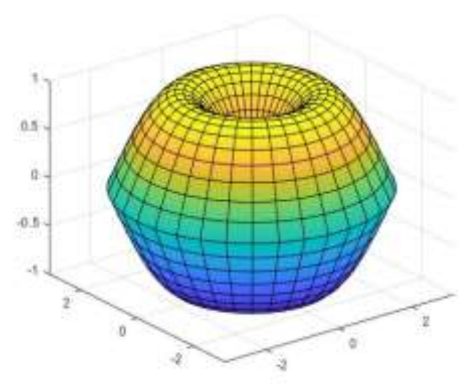

Figure 16: Graph of M_6

4.6. Case 6. If we put $\psi(u) = sin\ u$, we denote this surface by M_6 (see Fig. 16).
In view of (3.14-3.22), we have the following corollaries:

(i) M_6 is Lw-surface if the following equation is valid

$$m_1(u^{-1}cos\ u - sin\ u) - 2m_2 = 0. \tag{4.17}$$

The numerical solution of this equation when m_1=3, $m_2 = 4$ is $u = 848/2661$.
The Eq. (4.17) and its solution are illustrated in (Fig. 17).

(ii) M_6 is bi-conservative if the following equation is valid

$$(3sin\ u - u^{-1}cos\ u)(u^{-1}sin\ u + (u^{-2} + 1)cos\ u) = 0. \tag{4.18}$$

The numerical solution of this equation is $u = -5058/23$.

The Eq. (4.18) and its numerical solution are illustrated in (Fig. 18).

(iii) M_6 is harmonic if the following equation is valid.

$$\cos u - u\ sin\ u = 0. \tag{4.19}$$

The numerical solution of this equation is $u = -49990/221$.

The Eq. (4.19) and its numerical solution are illustrated in (Fig. 19).

(iv) M_6 is bi-harmonic if the following polynomials are valid

$$a_1 = w_6\ \ cos\ v = 0,\ \ a_2 = w_6\ \ sin\ v = 0, \tag{4.20}$$

$$a_3 = u(u^2 + 1)sin\ u - (2u^2 - 1)cos\ u = 0, \tag{4.21}$$

where w_6 is given by

$$w_6 = 3u^{-1}(sin^2 u - cos^2 u) + (3u^{-2} + 4)sin\ u\ \ cos\ u. \tag{4.22}$$

The Eqs. (4.20) and (4.21) are illustrated in (Fig. 20).

(v) M_6 is stable iff the following equation is valid

$$(cos\ u - usin\ u)^3 - 2u^3 sin\ u + 4u^2 cos\ u - 2usin\ u - 2cos\ u = 0. \tag{4.23}$$

The numerical solution of this equation is $u = -13346/59$.
The Eq. (4.23) and its numerical solution are illustrated in (Fig. 21).

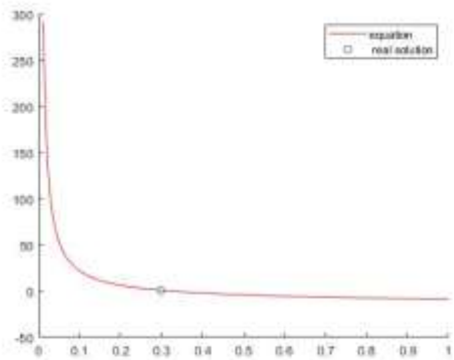

Figure 17: Lw-surface of M_6

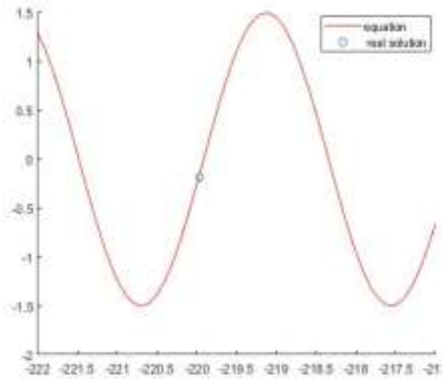

Figure 18: Bi-con. of M_6

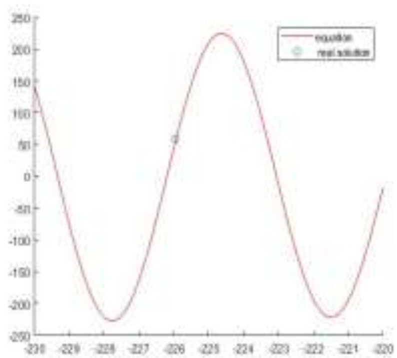

Figure 19: Harmonic of M_6

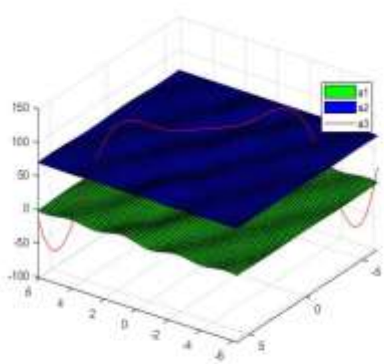

Figure 20: Bi-harmonic of M_6

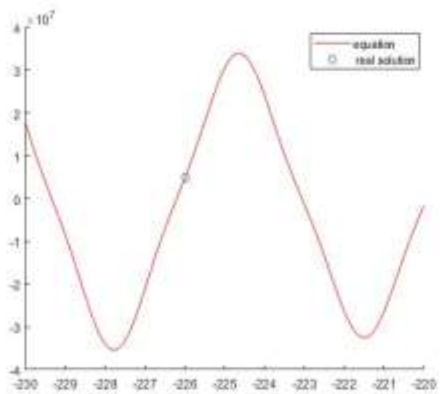

Figure 21: Stability of M_6

5. CONCLUSION

This work simply provided an approach to study revolution surfaces in a new form. General properties of these surfaces are obtained. We find that choosing different forms of the function $\psi(u)$ resulted in many revolution surfaces which have many uses in our daily life as in the figures No. (1, 2, 4, 9, 10 and 16). We were able to translate the basic equations of LW, bi-conservative, harmonic and stability revolution surfaces in the form of curves and then clarify the solutions of these equations theoretically and numerically, which are the real roots of those equations in the form of circles located on those curves as in the figures No. (3, 6, 8, 12, 13, 15, 17 and 18). As for the bi-harmonic condition, which was split into three conditions, the first two conditions denoted by a_1 and a_2 as a functions of the local coordinate on M, could be translated as two surfaces in 3-dimensional space, and the third condition denoted by a_3 as a function of one variable u that represents a curve in the plane as in the figures No. (7, 14 and 20).

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