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Semi-Baer and Semi-Quasi Baer Properties of Skew Generalized Power Series Rings

R. E. Abdel-Khalek [1], M. M. Hamam [2,*] and R. M. Salem [1]

[1] Mathematics Department, Faculty of Science, Al-Azhar Univ., Nasr City 11884, Cairo, Egypt.

[2] Mathematics Department, Faculty of Science, Assiut Univ., Assiut 71515, Egypt.

[*]Corresponding Author: mostafahamam154@gmail.com

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ABSTRACT

Let R be a ring with identity, $(S, \leq)$ an ordered monoid, $\omega: S \to End(R)$ a monoid homomorphism, and $A = R[[S, \omega]]$ the ring of skew generalized power series. The concepts of semi-Baer and semi-quasi Baer rings were introduced by Waphare and Khairnar as extensions of Baer and quasi-Baer rings, respectively. A ring R is called a semi-Baer (semi-quasi Baer) ring if the right annihilator of every subset (right ideal) of R is generated by a multiplicatively finite element in R. In this paper, we examine the behavior of a skew generalized power series ring over a semi-Baer (semi-quasi Baer) ring and prove that, under specific conditions, the ring A is semi-Baer (semi-quasi Baer) if and only if R is semi-Baer (semi-quasi Baer). Also, we prove that if f is a multiplicative finite element of A, then $f\,(1)$ is a multiplicative finite element of R and determine the conditions under which $f = c_{f(1)}$.

1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity, and $r_R(S) = \{a \in R \mid sa = 0,$ for all $s \in S\}$ is the right annihilator of a nonempty subset S in R. The notion of Baer rings was introduced by Kaplansky in 1955 [8]. Five years later, Maeda [12] defined Rickart rings and Hattori [6] introduced the notion of a right PP rings and it was later shown that the two concepts are equivalent. According to Chase [4] the PP ring notion is not left-right symmetric. Later in 1967 Clark introduced the concept of quasi-Baer rings [5]. By virtue of [8, Theorem 3] and [5, Lemma 1], the definitions of Baer and quasi-Baer rings are left-right symmetric. Hence, Brikenmeier introduced the concept of right principally quasi-Baer (right p.q. Baer) rings as a generalization of quasi-Baer rings (see [3]). The notion of a right p.q. Baer ring is not left–right symmetric, but if R is a semiprime ring, R is right p.q. Baer if and only if R is left p.q. Baer [3, Corollary 1.11].

In 2004, Lee and Zhou [10] introduced the concepts of Baer modules, PP modules, quasi-Baer modules, and p.q. Baer modules as extensions of Baer rings, PP rings, quasi-Baer rings, and p.q. Baer rings, respectively. As a generalization of Lee and Zhou's concepts, Waphare and Khairnar [26] introduced the notions of semi-Baer modules, semi-PP modules, semi-quasi-Baer modules, and semi-p.q. Baer modules.

In 1974, Armendariz seems to be the first to consider how a polynomial ring behaves over a Baer ring by proving that: If R is a reduced ring, then $R[x]$ is a Baer ring if and only if R is a Baer ring [2, Theorem B]. In 2005, Salem investigated how the generalized power series ring behaves over a Baer ring and showed that, under certain conditions the ring of generalized power series $A = R[[S]]$ is Baer if and only if R is Baer [25, Theorem 3.5]. Paykan and Moussavi extended these results by studying the relation between the Baer (quasi-Baer) properties of a ring R, and its skew generalized power series extension $R[[S, \omega]]$ [18, Theorem 2.11 and Theorem 2.17].

Inspired by those previous works, we examine the behavior of a skew generalized power series ring over a semi-Baer (semi-quasi Baer) ring and investigate the conditions under which a ring of skew generalized power series $R[[S,\omega]]$ is semi-Baer (semi-quasi Baer) whenever R is semi-Baer (semi-quasi Baer) and vice versa. Also, in the context of multiplicatively finite elements we generalize [17, Proposition 3.2].

2. Preliminaries

In this section, we present some definitions and results that will be helpful in the sequel.

Definition 2.1 ([8]). A ring R is called Baer if the right annihilator of every nonempty subset of R is generated by an idempotent.

Definition 2.2 ([5]). A ring R is called quasi-Baer if the right annihilator of every right ideal of R is generated by an idempotent.

In [26], Waphare and Khairnar introduced the concepts of a multiplicative order of an element and a multiplicatively finite element in rings as follows:

Definition 2.3. Let R be a ring with identity, $a \in R$ a nonzero element, and $S = \{a^t \mid 0 \leq t < \infty\}$. If S is finite, then the smallest positive integer k such that $a^k = a^m$ for some $0 \leq m < k$ is called a multiplicative order of the element a. If S is infinite, then the multiplicative order of a is infinity.

An element $a \in R$ is called multiplicatively finite if the multiplicative order of a is finite. Consequently, idempotents of a ring are multiplicatively finite elements of order 2. We assume multiplicative order of 0 to be 0.

Definition 2.4 ([26]). A ring R is called a semi-Baer (semi-quasi Baer) ring if the right annihilator of every subset (right ideal) of R is generated by a multiplicatively finite element in R.

Clearly, every Baer (quasi-Baer) ring is semi-Baer (semi-quasi Baer). However, the converse is true if the ring is reduced (see [26, Theorem 2.4]).

Lemma 2.5 ([26, Lemma 2.1]). An element $b \in R$ is multiplicatively finite if and only if there exists $i \in \mathbb{N}$ such that $(b^i)^2 = b^i$.

Further properties of semi-Baer (semi-quasi Baer) rings can be found in [16] and [26].

Definition 2.6 ([1]). An endomorphism σ of a ring R is called compatible if for all $a, b \in R,\ ab = 0$ if and only if $a\,\sigma(b) = 0$.

Definition 2.7 ([9]). An endomorphism σ of a ring R is called rigid if for every $a \in R$, $a\,\sigma(a) = 0$ if and only if $a = 0$.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. As in [14], a ring R is S-compatible (S-rigid) if ω_s is compatible (rigid) for every $s \in S$.

3. Skew Generalized Power Series Rings

The construction of generalized power series rings was considered by Higman in [7]. Paulo Ribenboim studied extensively in a series of papers (see [20]-[24]) the rings of generalized power series. In [15] Mazurek and Ziembowski generalized this construction by introducing the concept of the skew generalized power series rings.

An ordered monoid is a pair $(S, \leq)$ consisting of a monoid S and a compatible order relation $\leq$ such that if $u \leq v$, then $ut \leq vt$ and $tu \leq tv$ for each $t \in S$. $(S, \leq)$ is called a strictly ordered monoid if whenever $u, v \in S$ such that $u < v$ (i.e., $u \leq v$ and $u \neq v$), then $ut < vt$ and $tu < tv$ for all $t \in S$. Recall that an ordered set $(S, \leq)$ is called artinian if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid, $\omega: S \to End(R)$ a monoid homomorphism, where ω_s denote the image of s under ω, for each $s \in S$, that is $\omega_s = \omega(s)$, and A the set of all maps $f : S \to R$ such that $supp\,(f) = \{s \in S : f(s) \neq 0\}$ is an artinian and narrow subset of S. Under pointwise addition A is an abelian subgroup of the additive group of all mappings $f : S \to R$. For every $s \in S$ and $f, g \in A$

the set $X_s(f,g) = \{(u,v) \in S \times S: uv = s, f(u) \neq 0, g(v) \neq 0\}$ is finite by [21, 4.1]. Define the multiplication for each $f, g \in A$ by:

$fg(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))$. (By convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring called the ring of skew generalized power series whose elements have coefficients in R and exponents in S. For each $r \in R$ and $s \in S$ one can associate the maps $c_r, e_s \in A$ defined by:

$$c_r(x) = \begin{cases} r & if\ x = 1_s \\ 0 & otherwise \end{cases}, \quad e_s(x) = \begin{cases} 1_R & if\ x = s \\ 0 & otherwise \end{cases}$$

It is clear that $r \to c_r$ is a ring embedding of R into A and $s \to e_s$ is a monoid embedding of S into the multiplicative monoid of A and $e_s c_r = c_{\omega_s(r)} e_s$. Moreover, the identity element of A is a map $e : S \to R$ defined by $e(1_s) = (1_R)$ and $e(s) = 0$ for each $s \in S\backslash\{1_s\}$.

Let R be a ring and σ an endomorphism of R. The construction of the skew generalized power series rings generalizes many classical ring constructions such as the skew polynomial rings $R[x,\sigma]$ if $S = N \cup \{0\}$ and $\leq$ is the trivial order, skew power series rings $R[[x,\sigma]]$ if $S = N \cup \{0\}$ and $\leq$ is the natural linear order, skew Laurent polynomial rings $R[x,x^{-1};\sigma]$ if $S = Z$ and $\leq$ is the trivial order where σ is an automorphism of R, skew Laurent power series rings $R[[x,x^{-1};\sigma]]$ if $S = Z$ and $\leq$ is the natural linear order where σ is an automorphism of R. Moreover, the ring of polynomials $R[x]$, the ring of power series $R[[x]]$, the ring of Laurent polynomials $R[x,x^{-1}]$, and the ring of Laurent power series $R[[x,x^{-1}]]$ are special cases of the skew generalized power series rings, if we consider σ to be the identity map of R.

Recall that a ring R is said to be (S,ω)-Armendariz if whenever $fg = 0$ for $f, g \in R[[S,\omega]]$, then $f(s).\omega_s(g(t)) = 0$ for all $s, t \in S$ (see [14, Definition 2.1]). If we let ω to be the identity homomorphism, then R is said to be S-Armendariz if whenever $f, g \in R[[S]]$ (the ring of Generalized power series) satisfy $fg = 0$, then $f(u)g(v) = 0$ for each $u, v \in S$ (see [11]). If we let $S = N$ and the order $\leq$ is trivial, then R is said to be

Armendariz if whenever polynomials $f(x)=\sum_{i=0}^{m} a_i x^i$, $g(x)=\sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x)=0$, then $a_i b_j = 0$ for every i and j (see [19]).

4. Main Results

For an element $f \in R[[S,\omega]]$, let $\pi(f)$ denote the set of minimal elements of $supp\,(f)$. If $(S,\leq)$ is totally ordered, then $\pi(f)$ consists of only one element.

Definition 4.1([13]). An ordered monoid $(S,\leq)$ is said to be quasitotally ordered (and $\leq$ is called a quasitotal order on S) if $\leq$ can be refined to an order $\preccurlyeq$ with respect to which S is a strictly totally ordered monoid.

An ordered monoid $(S,\leq)$ is called positively ordered if 1 is the minimal element of S.

Lemma 4.2. Let R be a ring, $(S,\leq)$ a positively quasitotally ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Set $A = R[[S,\omega]]$ the ring of skew generalized power series. If f is a multiplicative finite element of A, then $f(1)$ is a multiplicative finite element of R.

Proof. Since f is a multiplicative finite element of $A = R[[S,\omega]]$, there exists $n \in N$ such that $((f)^n)^2 = (f)^n$ (see Lemma 2.5). By hypothesis, the order $\leq$ can be refined to a strict total order $\preccurlyeq$ on S. Consequently, there exists $s_0 \in supp(f)$ such that s_0 is a minimal element of $supp(f)$ under the total order $\preccurlyeq$. If $s_0 \neq 1$, then $s_0 \succ 1$ and $f(1)=0$ which implies that $((f(1))^n)^2 = (f(1))^n = 0$. That is $f(1)$ is a multiplicative finite element of R. If $s_0 = 1$, then we have $(f)^n(1) = ((f)^n)^2(1)$ where,

$$(f)^n(1) = \sum_{(u_1,u_2,\dots u_n)\in X_1(f,f,\dots,f)} f(u_1)\omega_{u_1}\big(f(u_2)\big)\omega_{u_1u_2}\big(f(u_3)\big)\dots\,\omega_{u_1u_2\dots u_{n-1}}(f(u_n)).$$

If at least one of $u_i \succ 1$, then $1 = u_1.u_2.u_3 \dots u_i \dots u_n \succ 1$ which is a contradiction. Thus, $u_1 = u_2 = u_3 \dots = u_n = 1$. Therefore,

$$(f)^n(1) = f(1)\omega_1\big(f(1)\big)\omega_1\big(f(1)\big)\dots\,\omega_1(f(1)).$$

Since $\omega: S \to End(R)$ is a monoid homomorphism, then we get $(f)^n(1) = \big(f(1)\big)^n$. Hence $\big(f(1)\big)^n = ((f)^n)^2(1) = (\big(f(1)\big)^n)^2$. That is $f(1)$ is a multiplicative finite element of R.

Proposition 4.3. Let R be a ring, $(S, \leq)$ a positively quasitotally ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Set $A = R[[S, \omega]]$ the ring of skew generalized power series.

(1) If A is semi-Baer, then R is semi-Baer.

(2) If R is an S-compatible ring and A is semi-quasi Baer, then R is semi-quasi Baer.

Proof. (1) Let X be a non-empty subset of R. Then $B = \{c_x : x \in \mathrm{X}\}$ is a non-empty subset of A. Since A is semi-Baer, there exists $f \in A$ such that $r_A(B) = fA$ with $((f)^n)^2 = (f)^n \, for\ some\ n \in N$. Since $(S, \leq)$ is a quasitotally ordered monoid, the order $\leq$ can be refined to a strict total order $\preccurlyeq$ on S. Lemma 4.2 implies that $f(1)$ is a multiplicative finite element of R. We want to prove that $r_R(X) = f(1)R$. Since $f \in r_A(B)$, then $c_x\, f = 0$ for all $c_x \in B$. Thus $0 = (c_x\, f)\,(1) = c_x(1)\, \omega_1\big(f(1)\big) = c_x(1) f(1) = \, x\, f(1)$ for all $x \in X$. Hence $f(1) \in r_R(X)$, which implies that $f(1)R \subseteq r_R(\mathrm{X})$. On the other hand, if $a \in r_R(X)$, then $(c_x c_a)(1) = c_x(1)\omega_1(c_a(1)) = c_x(1) c_a(1) = xa = 0$ for all $x \in X$. Thus $c_x c_a = 0$ for all $x \in X$ which implies that $c_a \in r_A(B)$ and $c_a = fg$ for some $g \in A$. Now, $a = c_a(1) = (fg)(1) = f(1)\, \omega_1(g(1)) \in f(1)R$. That is $r_R(X) \subseteq f(1)R$, which follows that $r_R(X) = f(1)R$. Hence, R is a semi–Baer ring.

(2) Let I be a right ideal of R. Then $I\big[[S, \omega]\big] = \{f \in A \mid f(s) \in I$ for any $s \in S\}$ is a right ideal of A. Since A is semi-quasi Baer, there exists $f \in A$ such that $r_A(I[[S, \omega]]) = fA$ with $((f)^n)^2 = (f)^n$ for some $n \in N$. Since $(S, \leq)$ is a quasitotally ordered monoid, the order $\leq$ can be refined to a strict total order $\preccurlyeq$ on S. Lemma 4.2 implies that $f(1)$ is a multiplicative finite element of R. We want to prove that $r_R(I) = f(1)R$. Since $f \in r_A(I[[S, \omega]])$, then $g\, f = 0$ for all $g \in I[[S, \omega]]$. Since $c_x \in$

$I[[S,\omega]]$ for all $x \in I$, we have $c_x f = 0$. Consequently, $(c_x f)(1) = 0$ which implies that $x\,f(1) = 0$ for all $x \in I$. Hence, $f(1) \in r_R(I)$, which implies that $f(1)R \subseteq r_R(I)$. On the other hand, if $a \in r_R(I)$, then $i\,a = 0$ for all $i \in I$. Since $g(s) \in I$ for all $g \in I[[S,\omega]]$ and $s \in S$, we have $g(s)\,a = 0$. Since R is S-compatible, we have $(gc_a)(s) = g(s)\omega_s(c_a(1)) = g(s)\omega_s(a) = 0$ for all $s \in S$. which implies that $c_a \in r_A(I[[S,\omega]])$ and $c_a = fg$ for some $g \in A$. Now, $a = c_a(1) = (fg)(1) = f(1)\,\omega_1(g(1)) \in f(1)R$. That is $r_R(I) \subseteq f(1)R$, which follows that $r_R(I) = f(1)R$. Hence, R is a semi-quasi Baer ring.

Proposition 4.4. Let R be a S-compatible (S,ω) Armendariz ring, $(S\,,\leq)$ a quasitotally ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Set $A = R[[S,\omega]]$ the ring of skew generalized power series.

(1) If R is a semi-Baer ring, then A is semi-Baer.

(2) If R is a semi-quasi Baer ring, then A is semi-quasi Baer.

Proof. (1) Let B be a non-empty subset of A. Then $\mathrm{U} = \{f(\mathrm{s}) : f \in B\,,\ s \in S\}$ is a non-empty subset of R. Since R is semi-Baer, there exists $b \in R$ such that $r_R(\mathrm{U}) = bR$ with $((b)^n)^2 = (b)^n$ for some $n \in N$. Therefore, c_b is a multiplicatively finite element of A. We want to prove that $r_A(B) = c_b A$. Since $b \in r_R(\mathrm{U})$, it follows that $f(s)b = 0$ for all $f(s) \in \mathrm{U}$. Thus, $f(s)c_b(1) = 0$. Since R is S-compatible, then $(fc_b)(s) = f(s)\omega_s(c_b(1)) = 0$ for all $s \in S$. Thus $c_b \in r_A(B)$ which implies that $c_b A \subseteq r_A(B)$. Now, let $f \in r_A(B)$. Then $gf = 0$ for all $g \in B$. Since R is a (S,ω) Armendariz ring, we get $g(u)\omega_u f(v) = 0$ for all $u, v \in S$. Moreover, since R is S-compatibe, we have $g(u)f(v) = 0$ for all $u, v \in S$. Thus $f(v) \in r_R(\mathrm{U}) = bR$ for all $v \in S$. Therefore, for all $v \in S$ there exists $r \in R$ such that $f(v) = br = (c_b c_r e_v)(v)$. Thus $f = c_b c_r e_v$, which implies that $f \in c_b A$. That is $r_A(B) \subseteq c_b A$, which follows that $r_A(B) = c_b A$. Hence, A is a semi-Baer ring.

(2) Let J be a right ideal of A. For every $s \in S$, set $J_s = \{f(\mathrm{s}) : f \in J\,,\ s \in S\}$, and $J^* = \cup_{s\in S} J_s$. Let I be the right ideal generated by J^*. Since R is semi-quasi Baer, there exists $b \in R$ such that $r_R(I) = bR$ with $((b)^n)^2 = (b)^n$ for some $n \in N$. Therefore, c_b

is a multiplicatively finite element of A. We want to prove that $r_A(J) = c_b A$. Since $b \in r_R(I)$, we have $i\, b = 0$ for all $i \in I$. Since $g(s) \in I$ for all $g \in J$ and $s \in S$, we have $g(s)\, b = 0$. Thus $g(s)c_b(1) = 0$. Since R is S-compatibe, we have $(gc_b)(s) = g(s)\omega_s(c_b(1)) = 0$ for all $s \in S$. Thus $c_b \in r_A(J)$ which implies that $c_b A \subseteq r_A(J)$. Now, let $f \in r_A(J)$. Then $gf = 0$ for all $g \in J$. Since R is a (S, ω) Armendariz ring, we get $g(u)\omega_u(f(v)) = 0$ for all $u, v \in S$. Moreover, since R is S-compatibe, then we have $g(u)f(v) = 0$ for all $u, v \in S$. Thus $f(v) \in r_R(I) = bR$ for all $v \in S$. Therefore, for all $v \in S$ there exists $r \in R$ such that $f(v) = br = (c_b c_r e_v)(v)$. Thus $f = c_b c_r e_v$, which implies that $f \in c_b A$. That is $r_A(J) \subseteq c_b A$, which follows that $r_A(J) = c_b A$. Hence, A is a semi-quasi Baer ring.

From Proposition 4.3 and Proposition 4.4 we have the following:

Theorem 4.5. Let R be a S-compatible (S, ω)Armendariz ring, $(S, \leq)$ a positively quasitotally ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Set $A = R[[S, \omega]]$ the ring of skew generalized power series. Then A is semi-Baer (semi-quasi Baer) ring if and only if R is semi-Baer (semi-quasi Baer).

Corollary 4.6. Let R be a S-Armendariz ring and $(S, \leq)$ a positively quasitotally ordered monoid. Set $A = R[[S]]$ the ring of generalized power series. Then A is semi-Baer (semi-quasi Baer) ring if and only if R is semi-Baer (semi-quasi Baer).

Corollary 4.7. Let R be an Armendariz ring. Then $R[x]$ and $R[[x]]$ are semi-Baer (semi-quasi Baer) rings if and only if R is semi-Baer (semi-quasi Baer).

The following result generalize [17, Proposition 3.2].

Proposition 4.8. Let R be a ring, $(S, \leq)$ a quasitotally ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Assume that R is S-rigid. If f is a multiplicative finite element of $R[[S, \omega]]$, then $f(1)$ is a multiplicative finite element of R and $f = c_{f(1)}$.

Proof. Since f is a multiplicative finite element of $R[[S, \omega]]$, there exists $n \in N$ such that $((f)^n)^2 = (f)^n$ (see Lemma 2.5). Since the order $\leq$ can be refined to a strict total order $\preccurlyeq$ on S, it follows that there exists $u_0 \in supp(f)$ such that u_0 is a minimal element of $supp(f)$ under the total order $\preccurlyeq$ which implies that u_0^n is the minimal element of $supp(f^n)$. For any $(u,v) \in X_{(u_0^n)^2}(f^n, f^n)$, $u_0^n \preccurlyeq u$, $u_0^n \preccurlyeq v$. If $u_0^n \prec u$, since $\preccurlyeq$ is a strict order, $(u_0^n)^2 \prec u\, u_0^n \preccurlyeq u\, v = (u_0^n)^2$, a contradiction. Thus $u = u_0^n$. similarly, $v = u_0^n$. Hence

$$(f^n)^2(u_0^n)^2 = \sum_{(u,v)\in X_{(u_0^n)^2}(f^n,f^n)} f^n(u)\omega_u(f^n(v)) = f^n(u_0^n)\omega_{u_0^n}(f^n(u_0^n)). \qquad (4.1)$$

Assume that $u_0 \prec 1$. Since $\preccurlyeq$ is a strict order relation, it follows that $(u_0^n)^2 \prec u_0^n$. Hence, the minimality of $supp(f^n)$ implies that $f^n(u_0^n)^2 = 0$. From $((f)^n)^2 = (f)^n$ and equation (4.1) we infer that $f^n(u_0^n)\omega_{u_0^n}(f^n(u_0^n)) = 0$. Since R is S-rigid, we obtain $f^n(u_0^n) = 0$, which contradicts the fact that u_0^n is a minimal element of $supp(f^n)$. Hence, $u_0 \succcurlyeq 1$.

Suppose that there exists $s_0 \succ 1$ such that $f^n(s_0^n) \neq 0$. Assume that s_0^n is the smallest with the condition under the total order $\preccurlyeq$. Therefore, $f^n(s) = 0$ for all $1 \prec s \prec s_0^n$. From $((f)^n)^2 = (f)^n$, it implies that

$$((f)^n)^2(1) = (f)^n(1) \text{ and } (f)^n(s_0^n) = (f)^n(s_0^n)\omega_{s_0^n}((f)^n(1)) + (f)^n(1)(f)^n(s_0^n).$$

Since $(f)^n(1)$ is an idempotent element of the ring R, from [17, lemma 3.1(1)], we infer

$$(f)^n(s_0^n) = (f)^n(s_0^n)(f)^n(1) + (f)^n(1)(f)^n(s_0^n). \qquad (4.2)$$

Multiplying equation (4.2) on the left by $(f)^n(1)$ we have

$$(f)^n(1)(f)^n(s_0^n) = (f)^n(1)(f)^n(s_0^n)(f)^n(1) + (f)^n(1)(f)^n(s_0^n).$$

Thus $(f)^n(1)(f)^n(s_0^n)(f)^n(1) = 0$. Multiplying on the left by $(f)^n(s_0^n)$ we get $((f)^n(s_0^n)(f)^n(1))^2 = 0$ and since R is reduced, $(f)^n(s_0^n)(f)^n(1) = 0$. Substituting in equation (4.2) we get $(f)^n(s_0^n) = 0$, which is a contradiction. Consequently, we have $f(s) = 0$ for all $s \in S\backslash\{1\}$. Thus $f = c_{f(1)}$ as desired.

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